LABORATORY MANUAL

DEPARTMENT OF
ELECTRICAL & COMPUTER ENGINEERING

UNIVERSITY OF CENTRAL FLORIDA

EEL 3552
Signal Analysis & Communications

Revised
UCF Valencia West
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Safety Rules and Operating Procedures

1. Note the location of the Emergency Disconnect (red button near the door) to shut off power in an emergency. Note the location of the nearest telephone (map on bulletin board).

2. Students are allowed in the laboratory only when the instructor is present.

3. Open drinks and food are not allowed near the lab benches.

4. Report any broken equipment or defective parts to the lab instructor.
   Do not open, remove the cover, or attempt to repair any equipment.

5. When the lab exercise is over, all instruments, except computers, must be turned off.
   Return substitution boxes to the designated location. Your lab grade will be affected if your laboratory station is not tidy when you leave.

6. University property must not be taken from the laboratory.

7. Do not move instruments from one lab station to another lab station.

8. Do not tamper with or remove security straps, locks, or other security devices.
   Do not disable or attempt to defeat the security camera.

9. ANYONE VIOLATING ANY RULES OR REGULATIONS MAY BE DENIED ACCESS TO THESE FACILITIES.

I have read and understand these rules and procedures. I agree to abide by these rules and procedures at all times while using these facilities. I understand that failure to follow these rules and procedures will result in my immediate dismissal from the laboratory and additional disciplinary action may be taken.

_____ _____

Signature Date Lab #

Laboratory Safety Information

Introduction

The danger of injury or death from electrical shock, fire, or explosion is present while conducting experiments in this laboratory. To work safely, it is important that you understand the prudent practices necessary to minimize the risks and what to do if there is an accident.

Electrical Shock

Avoid contact with conductors in energized electrical circuits. Electrocution has been reported at dc voltages as low as 42 volts. 100ma of current passing through the chest is usually fatal. Muscle contractions can prevent the person from moving away while being electrocuted.

Do not touch someone who is being shocked while still in contact with the electrical conductor or you may also be electrocuted. Instead, press the Emergency Disconnect (red button located near the door to the laboratory). This shuts off all power, except the lights.

Make sure your hands are dry. The resistance of dry, unbroken skin is relatively high and thus reduces the risk of shock. Skin that is broken, wet, or damp with sweat has a low resistance.

When working with an energized circuit, work with only your right hand, keeping your left hand away from all conductive material. This reduces the likelihood of an accident that results in current passing through your heart.

Be cautious of rings, watches, and necklaces. Skin beneath a ring or watch is damp, lowering the skin resistance. Shoes covering the feet are much safer than sandals.

If the victim isn't breathing, find someone certified in CPR. Be quick! Some of the staff in the Department Office are certified in CPR. If the victim is unconscious or needs an ambulance, contact the Department Office for help or call 911. If able, the victim should go to the Student Health Services for examination and treatment.

Fire

Transistors and other components can become extremely hot and cause severe burns if touched. If resistors or other components on your proto-board catch fire, turn off the power supply and notify the instructor. If electronic instruments catch fire, press the Emergency Disconnect (red button). These small electrical fires extinguish quickly after the power is shut off. Avoid using fire extinguishers on electronic instruments.

Explosions

When using electrolytic capacitors, be careful to observe proper polarity and do not exceed the voltage rating. Electrolytic capacitors can explode and cause injury. A first aid kit is located on the wall near the door. Proceed to Student Health Services, if needed.

Laboratory 1
SPECTRUM ANALYSIS

OBJECTIVE:
Analyze the spectral content of a simple signal.

EQUIPMENT:

TDS5052B Oscilloscope
Function Generator
You should bring a floppy disk to store your waveforms.

BACKGROUND:

A waveform representing amplitude, as a function of time, is called a time domain display. It is also possible for a waveform to represent amplitude as a function of frequency. This is called a frequency domain display. A Spectrum Analyzer is an instrument, which can display the frequency domain of a signal. However, the TDS5052B Oscilloscope has the capability of producing both time domain and frequency domain displays (see Figure 1.1).

A sine wave is the simplest signal for spectral analysis. The amplitude of the sine wave can be determined on the vertical scale and the frequency can be determined on the horizontal scale.

The units of amplitude used in this experiment will be dBV, which is dB relative to 1 VRMS ($0\,dBV = 1\,VRMS$), according to the formula:

$$dBV = 20\log\left(\frac{Vsignal}{Vref}\right)$$

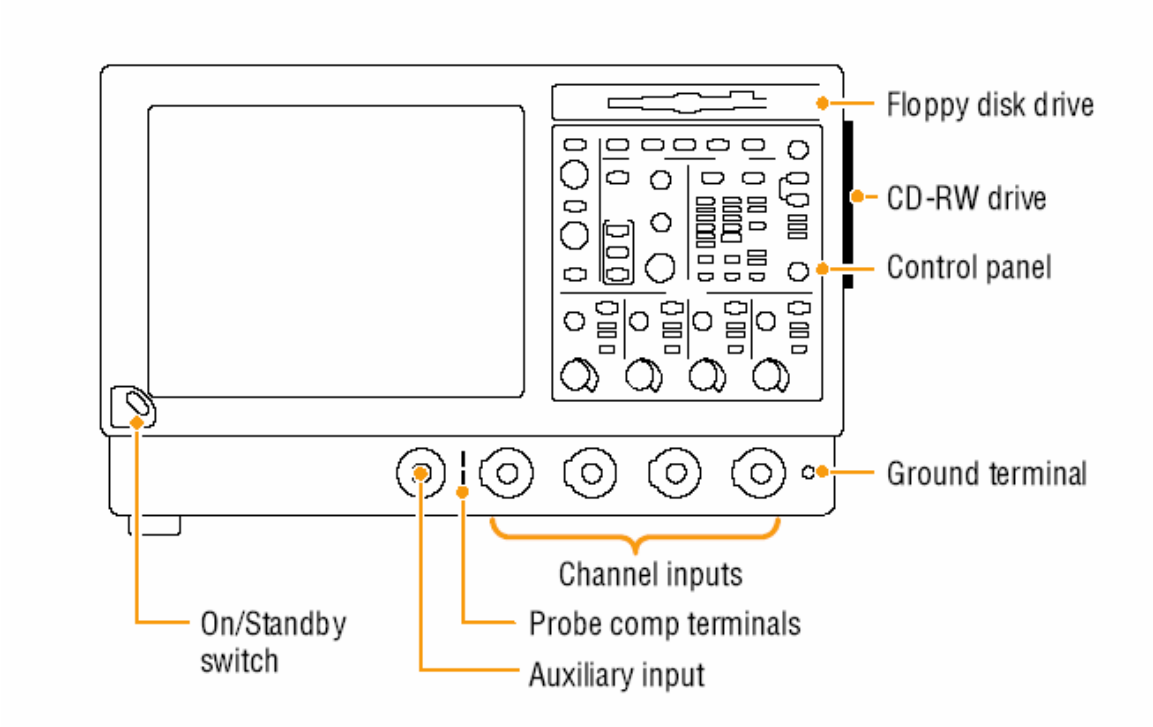
where $Vsignal$ is the RMS voltage of the signal and $Vref = 1$ volt RMS.

Another common unit of amplitude used for spectrum analysis is dBm, which is dB relative to 1 milliwatt ($0dBm = 1mW$), according to the formula:

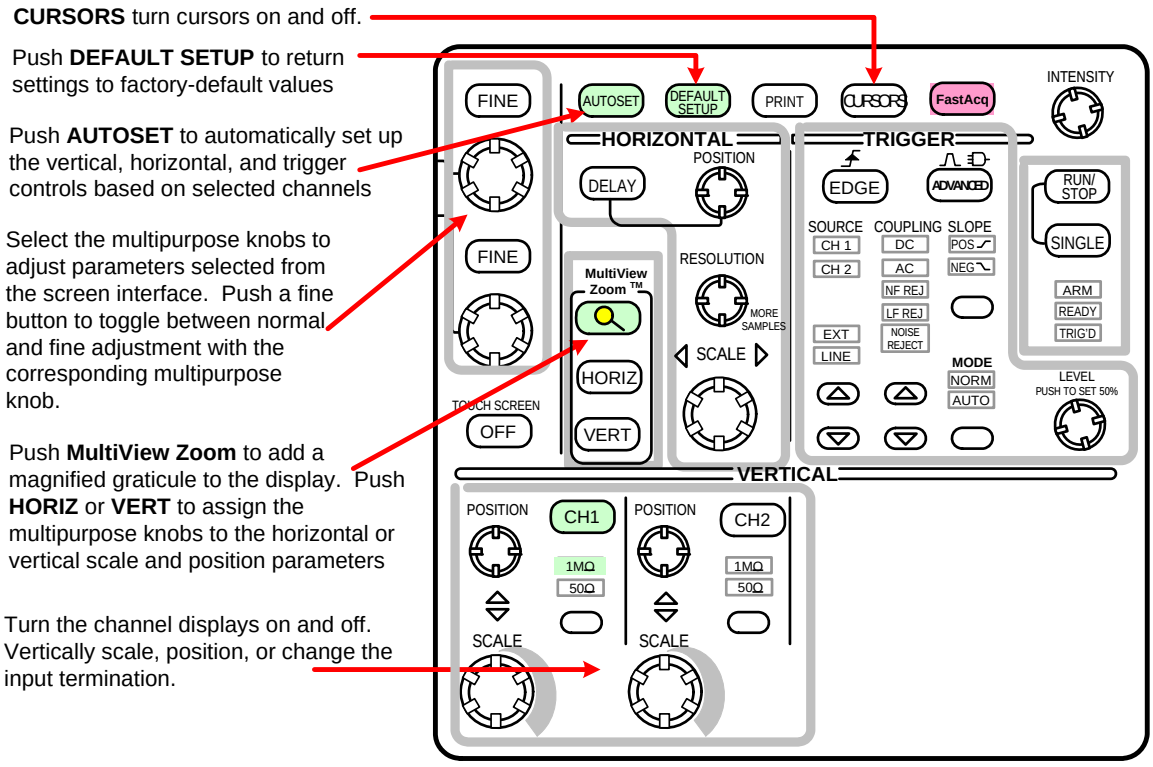
$$dBm = 10\log\left(\frac{Psignal}{1mW}\right)$$

where $Psignal$ is the power of the signal in milliwatts.

1. Front Panel of TDS 5052B



(a)



(b)

Figure 1.1 (a) Front Panel (b) Control Panel (Tektronix User Manual reference)

PREPATATION:

a. Calculate the amplitude in dBV of a 1 KHz 2 volt peak-to-peak sine wave.

b. Calculate the peak-to-peak voltage of a –10 dBV 4 KHz sine wave.

EXPERIMENT:

1. TIME DOMAIN DISPLAY

a. Turn on the Oscilloscope and allow a minute for the instrument to boot and stabilize. Then, press Default Setup to clear settings made by other students. Set the Vertical scale to 500 mv and the Horizontal scale to 1.0 ms per division.

b. Turn on the Function Generator and connect the output to Channel 1 input of the oscilloscope.

c. Set the Function Generator for a sine wave output with a frequency of 1 KHz. Adjust the amplitude to 2 volt peak-to-peak with zero DC offset. The Oscilloscope should now display 10 cycles of a sine wave. Do not change the Horizontal scale from 1.0 ms through the remainder of this experiment.

Note. Displaying fewer cycles in the time domain display would result in lower frequency resolution for the frequency domain display. Displaying less than 4 cycles would result in poor frequency resolution when the frequency domain of the sine wave is displayed.

2. FREQUENCY DOMAIN DISPLAY

a. Use the mouse to click the Math Button on the Toolbar and select Spectral Setup to open the Control Window.

b. Under the Create tab, click the Magnitude icon and click Channel 1 as source. Click Apply.

c. Click the Mag tab. Select dB for the vertical scale factor. Click the Reference Level Offset, set it to 1000 mV (1 Volt), and click enter. Click Apply.

d. In this experiment, a Gaussian window will be used, which is the default setting.

e. Click OK to close the Control Window.

f. Turn the time domain display off by pressing the CH1 button.

The frequency domain is now displayed. The orange M1 label on the left edge of the screen indicates the 0dB level. The orange label at the bottom left of the screen now says Math 1 20.0 dB 25.0KHz. This means each vertical division is 20dBV and each

horizontal division is 25.0 KHz. Thus, the frequency spectrum is displayed from 0 Hz (on the left) to 250 KHz on the right.

g. Press the Zoom icon that is located in the Multiview section of the Oscilloscope Control Panel. Zoom allows detailed analysis of a narrower frequency range.

The waveform in the upper portion of the screen displays the spectrum from 0 to 250 KHz and the lower waveform is the frequency range that is being expanded. A box appears in the upper waveform to identify the range that is being expanded. Adjust the upper Multipurpose knob to set the Position to 1% and adjust the lower Multipurpose knob to set the Factor to 50.
A Zoom Factor of 50 results in a display of

$$25KHz/50 = 500Hz \text{ per division.}$$

The white label at the bottom edge of the screen confirms that the Zoom display is set to 20 dB and 500 Hz per division.
Setting the position to 1%, sets the frequency at the center of the screen to
1% of 250 KHz = 2.5 KHz.
Thus, the Zoom is now set to display from 0 Hz to 5 KHz.
The 1 KHz signal should now be visible in the lower waveform.

h. Use the mouse to click the Cursors button on the toolbar. Select Cursors On. Click the Cursors button again, select Cursor Type, and then click Waveform. Two cursors appear in the upper waveform, both outside of the zoomed frequency range. Use the mouse to drag one of the cursors into the zoomed frequency range and position it at the peak of the 1 KHz signal. Does the amplitude agree with the value that you calculated in the Preparation?

i. Change the frequency of the Function Generator to 2 KHz. Does this accurately represent the input signal?
On the Toolbar, click File, click Save As, and save to your floppy disk.

Note. Do not (ever) save to the Local Disk (C:).

j. Change the frequency of the Function Generator to 4 KHz.
Does the frequency domain waveform confirm that the signal is 4 KHz?

k. Decrease the amplitude of the Function Generator output to –10 dBV as measured in the frequency domain display. Turn the time domain on again by pressing the CH1 button. Turn the Zoom off by pressing the Zoom icon. Read the peak-to-peak voltage measurement. Does this measurement agree with the value that you calculated in the Preparation?

REPORT:

In your report, describe what you have learned in this experiment. Compare your experimental measurements with the theoretical calculations. Remember to insert the picture that you saved as part of your report. Write all conclusions.

REFERENCES:

EXPERIMENT 2
Periodic Signal Spectra

1 Objective

To understand the relationships between time waveforms and frequency spectra.

2 Theory

2.1 Motivation

Consider for a moment the special class of systems that are called *linear time-invariant systems.* When a signal is applied to this system, the output can be expressed in terms of linear differential equations with constant coefficients. An example of such a system is shown in Figure 2.1. The differential equation that describes the input/output relationship in such a system is given below

$$\frac{d}{dt}v_0(t)+\frac{R}{L}v_0(t)=\frac{R}{L}v_i(t) \tag{2.1}$$

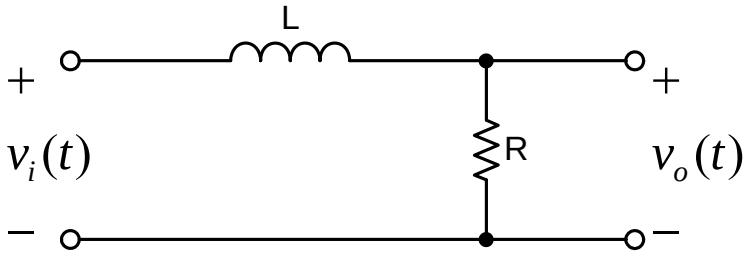


Figure 2.1 A Simple linear Time-invariant system

For a finite input $v_i(t)$ we expect to solve this differential equation to find the corresponding output $v_0(t)$. The difficulty involved in this task is that the input has to be a very simple signal. One way out from this complication is to express the input as a linear combination of simpler to deal with inputs. The choice of simpler inputs will affect the difficulty which we may encounter

in solving the problem.

There is an intriguing possibility regarding the simpler input signals. Let us for a moment assume that we choose as an input a function that repeats itself under the operation of differentiation. *We can show that such a function at the input will yield the same function as an output multiplied by some algebraic polynomial involving the parameters of the function and the parameters of the differential equation.*

A function that demonstrates this behavior, that is to repeat itself under the operation of differentiation, is the function e^{st}, where $s = \sigma + j\omega$ is a complex number. Since some of the signals that we are interested in are energy-type signals that may exist for positive or negative time it is wise to choose $\sigma = 0$. This discussion indicates that if the input signal $v_i(t)$ in equation (2.1) is of the form $e^{j\omega t}$, then we can readily find the output $v_0(t)$ of the system. Furthermore, if we can express the input signal $v_i(t)$ as a linear combination of complex exponential signals (of the form $e^{j\omega t}$), we can still find, without difficulty, the output $v_0(t)$ of the system (the reason being that the system in (2.1) is linear, and as a result if the input $(v_i(t))$ is a linear combination of complex exponentials ,then the output $(v_0(t))$ can be expressed as a linear combination of the outputs of the complex exponentials involved in the input). One version of the Fourier series expresses an arbitrary signal (like $v_i(t)$) as a linear combination of complex exponentials. This is the first reason that the Fourier series representation of a signal is so useful.

A close relative of a complex exponential $e^{j\omega t}$ are the signals $\cos\omega t$ and $\sin\omega t$ (remember Euler's identity: $e^{j\omega t} = \cos\omega t + j\sin\omega t$). Consider now the following set of familiar signals.

$$
\begin{aligned}
f_1(t) &= a\sin\omega_o t \\
f_2(t) &= a\sin 2\omega_o t \\
f_3(t) &= a\sin 3\omega_o t \\
&\cdot \quad\quad \cdot \\
&\cdot \quad\quad \cdot \\
&\cdot \quad\quad \cdot \\
f_n(t) &= a\sin n\omega_o t
\end{aligned}
\qquad (2.2)
$$

In Figure 2.2 we show a plot of $f_1(t)$ and $f_2(t)$. More generally, we can say that as n increase

the rate of variation of $f_n(t)$ increases as well.

One way to quantify the above qualitative statements is to take the derivative of $f_n(t)$ with respect to time. In particular,

$$\frac{d}{dt} f(t) = an\omega_0 \cos n\omega_0 t \tag{2.3}$$

From the above equation we observe that as *n* increases the maximum rate of variation of $f_n(t)$ with respect t time increases. We can represent each one of the $f_n(t)$ functions in a little bit different way than the one used in Figure 2.2. That is, we can represent these functions by plotting the maximum amplitude of $f_n(t)$ (i.e., a) versus the angular frequency of the sinusoid (i.e., $n\omega_0$). The product of these two values gives you the maximum rate of variation for the signal under consideration. In Figure 2.3 we are representing the functions $f_1(t)$, $f_2(t)$, and $f_3(t)$ following the aforementioned convention. Obviously, larger values for the location of the plotted amplitudes imply faster varying time signals; also the larger the amplitudes plotted at a particular location the faster the corresponding signals vary with respect to time.

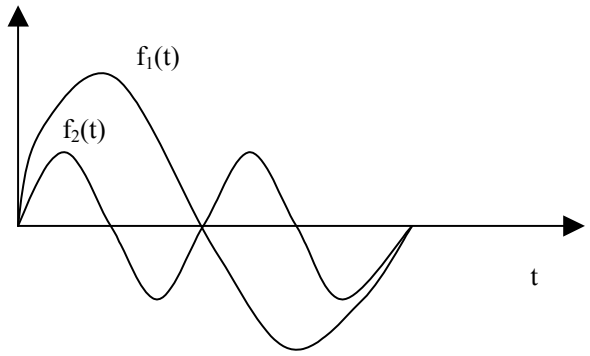


Figure 2.2 Plots of $f_1(t)$ and $f_2(t)$

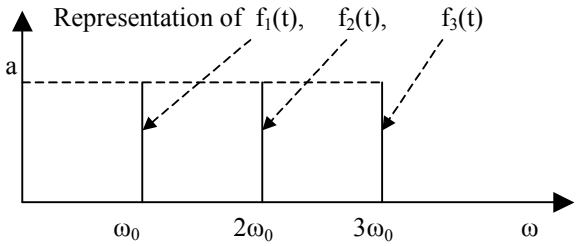


Figure 2.3 Reresentation of $f_1(t)$, $f_2(t)$, $f_3(t)$

Now consider a simple example of an amplitude-modulated signal.

$$f(t) = A(1 + \cos \omega_m t)\cos \omega_0 t \quad \omega_m << \omega_0 \tag{2.4}$$

In Figure 2.4 we plot the aforementioned signal *f(t)*. As we can see from the figure, the amplitude varies slowly between 0 and 2A. Its rate of variation is given by ω_m, the modulating frequency; ω_0 is referred to as the carrier frequency. The amplitude variations form the envelope of the complete signal, and represent any information being transmitted. Note that

$$A(1 + \cos \omega_m t)\cos \omega_0 t = A\cos \omega_0 t + \frac{A}{2}[\cos(\omega_0 - \omega_m)t + \cos(\omega_0 + \omega_m)t] \tag{2.5}$$

The representation of the aforementioned signal, using the new conventions, is illustrated in Figure 2.5. As we observe from Figure 2.5, larger carrier frequency corresponds to a faster varying signal; also larger modulating frequency results in a faster varying signal.

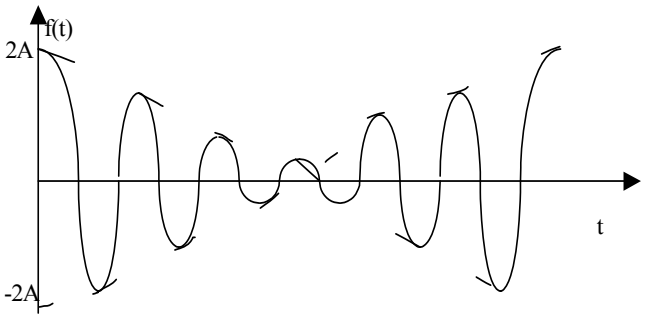


Figure 2.4 Plot of an amplitude-modulated signal f(t)

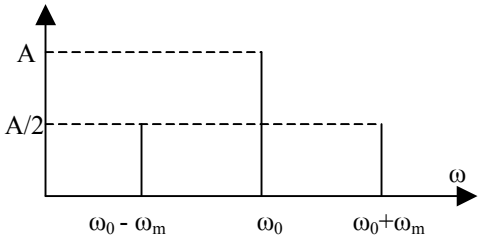


Figure 2.5 Representation of the amplitude-modulated signal f(t)

The previously discussed signals vary in some sinusoidal fashion. They do not carry any information but we have used them to focus on an alternative representation of the signal, that conveys to us the information of how fast the signal changes with respect to time. Another variation of the Fourier series representation of a signal represents the signal as a linear combination of sines and cosines of appropriate frequencies. As a result, the Fourier series representation of the signal, which is an extension of the alternative representation of the

simple signals that we discussed above, gives us an idea of how fast the signal changes with respect to time. This is the second reason that the Fourier series representation of a signal is so useful.

2.2 Fourier Series

Let our space S of interest be the set of piecewise continuous real time signals, defined over an interval $(0,T)$. Let us also consider the set of signals

$$\phi_{10}(t) = \frac{1}{\sqrt{T}} \tag{2.6}$$

$$\phi_{1n}(t) = \frac{2}{\sqrt{T}} \cos n\omega_0 t \tag{2.7}$$

$$\phi_{2n}(t) = \frac{2}{\sqrt{T}} \sin n\omega_0 t \quad n = 1,2,\cdots \tag{2.8}$$

where $\omega_0 = 2\pi/T$. The aforementioned signals (see equations (2.6), (2.7) and (2.8)) are orthonormal. They also constitute a complete set of functions. These two properties allow us to state that an arbitrary signal in S can be expressed as a linear combination of the signals in equations (2.6)-(2.8), such that

$$f(t) = C_{10}\phi_{10} + \sum_{n=1}^{\infty} C_{1n}\phi_{1n} + \sum_{n=1}^{\infty} C_{2n}\phi_{2n} \tag{2.9}$$

and the coefficients in the linear expansion above can be computed through the following equations:

$$C_{10} = \int_0^T f(t)\frac{1}{\sqrt{T}}dt$$

$$C_{1n} = \sqrt{\frac{2}{T}}\int_0^T f(t)\cos n\omega_0 t dt \quad n = 1,2,\cdots \tag{2.10}$$

$$C_{2n} = \sqrt{\frac{2}{T}}\int_0^T f(t)\sin n\omega_0 t dt \quad n = 1,2,\cdots$$

The above expansion, due to Fourier, is referred to as the *Trigonometric Fourier Series* expansion of a signal $f(t)$. A more common form for the trigonometric Fourier series expansion of a signal is given below.

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \tag{2.11}$$

where

$$a_0 = \frac{1}{T}\int_0^T f(t)dt \tag{2.12}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t dt \quad n = 1,2,\cdots \tag{2.13}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t dt \quad n = 1,2,\cdots \tag{2.14}$$

By a simple trigonometric manipulation we get that

$$f(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}\, \cos(n\omega_0 t + \theta_n) \quad \theta_n = \arctan\left\{\frac{-b_n}{a_n}\right\} \tag{2.15}$$

The above equation puts $f(t)$ in the desired form. In other words, we have now expressed $f(t)$ as a linear combination of sinusoids, and as a result we can represent this signal $f(t)$ by plotting the amplitude of each one of its sinusoids at the corresponding angular frequency location. This will give rise to the *amplitude — frequency* plot of $f(t)$. Observe though that in equation (2.15) each sinusoid involved in the expansion of $f(t)$ has a phase associated with it. Hence, in order to completely describe the signal $f(t)$ we need to draw another plot that provides the information of the phase associated with each sinusoid. This plot is denoted as the *phase — frequency* plot and it corresponds to a plot of θ_n at the location ω_n.

It is worth mentioning that the aforementioned expansion of $f(t)$ in terms of sinusoidal signals of frequencies that are integer multiples of the frequency ω_0 is often referred to as the expansion of $f(t)$ *in terms of its harmonic components.* The frequency ω_0 is called the *fundamental frequency,* or *first harmonic,* while multiples of the first harmonic frequency are referred to as the *second harmonic, third harmonic,* and so on. The corresponding signals in the expansion are named *first harmonic component, second harmonic component, third harmonic component,* and so on. The component a_0 in the above expansion is called the DC component of the signal $f(t)$.

Since we shall frequently be interested in the amplitudes, $\sqrt{a_n^2 + b_n^2}$'s, and phases, θ_n 's, of our signal $f(t)$ it would be simpler to obtain these directly from $f(t)$, rather than by first finding a_n and b_n. In order to achieve this, let us consider the functions

$$\phi_n(t) = \frac{1}{\sqrt{T}} e^{jn\omega_0 t} \tag{2.16}$$

where $\omega_0 = \frac{2\pi}{T}$ The above set of functions is a complete, orthonormal set. Therefore, any function $f(t)$, defined over an interval of length T, can be expanded as a linear combination of the $\phi_n(t)$'s. In particular,

$$f(t) = \sum_{n=-\infty}^{\infty} C_n \phi_n(t) = \sum_{n=-\infty}^{\infty} C_n \frac{1}{\sqrt{T}} e^{jn\omega_0 t} \tag{2.17}$$

The coefficients C_n in the above expansion are chosen according to the rules specified by the Orthogonality Theorem. That is,

$$C_n = \int_0^T f(t) \frac{1}{\sqrt{T}} e^{jn\omega_0 t} dt \tag{2.18}$$

The above equation is denoted as the *Exponential Fourier Series* expansion of the signal $f(t)$. Actually, in most instances, the exponential Fourier series expansion of a signal is provided by the following equation:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \tag{2.19}$$

where

$$F_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega_0 t} dt \tag{2.20}$$

Our original goal has now been accomplished, since we can now derive the amplitude — frequency plot of $f(t)$ by plotting the magnitude of F_n, with respect to frequency, and we can also derive the phase — frequency plot of $f(t)$by plotting the phase of F_n with respect to frequency. Note that, in general, the F_n's are complex numbers and a complex number F_n, can be written as the product of its magnitude ($|F_n|$) times $e^{j\angle F_n}$. One can show that the coefficients of the exponential Fourier series expansion of $f(t)$and the coefficients of the trigonometric Fourier series expansion of $f(t)$ are related as follows:

$$F_0 = a_0$$
$$F_n = \frac{1}{2}(a_n - jb_n) \quad 1 \le n \le \infty \tag{2.21}$$
$$F_{-n} = \frac{1}{2}(a_n + jb_n) \quad 1 \le n \le \infty$$

The following are some comments:

- The exponential Fourier series and the trigonometric Fourier series expansion were introduced for a signal $f(t)$ defined over the interval $(0,T)$. Actually, the formulas provided are valid for any signal $f(t)$ defined over any interval of length T (the starting and the ending points of they interval are immaterial, as far as the length of the interval is equal to T).

- We expanded so far a signal $f(t)$, defined over an interval of length T, in terms of a linear combination of sinusoids (Trigonometric Fourier Series expansion), or complex exponentials (Exponential Fourier Series expansion). It is worth noting, that the trigonometric or exponential Fourier series expansion of a signal defined over an interval T is a periodic signal of period T; that is it repeats itself with period T. Hence, if the signal of interest $f(t)$ is not periodic the trigonometric or exponential Fourier series expansion of the signal is only valid for the interval over which the signal is defined.

- Due to the periodicity nature of the Fourier series expansion, Fourier series is primarily used to represent signals of periodic nature. Signals of aperiodic nature can also be represented as a "sum" of sinusoids or complex exponentials, but this "sum" is in reality an integral and it is designated by the name *Fourier Transform.* The Fourier transform of an aperiodic signal will be introduced later as an extension of the Fourier series of a periodic signal.

2.3 Examples

2.3.1 Example 1

(a) Evaluate the trigonometric Fourier series expansion of a signal $f(t)$, *which* is depicted in Figure 2.6.

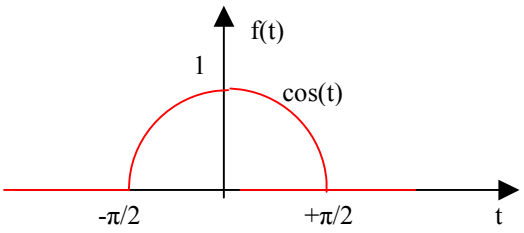


Figure 2.6 plot of f(t)=cost for [-π/2, π/2] and zero elsewhere

The interval of interest is $(-\pi/2,\pi/2)$. Hence $T=\pi$, and as a result $\omega_0 = 2\pi/T = 2$. The trigonometric Fourier series of a signal $f(t)$ is therefore of the form

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2nt + \sum_{n=1}^{\infty} b_n \sin 2nt \tag{2.22}$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos t dt = \frac{2}{\pi} \tag{2.23}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos 2ntdt = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \cos 2ntdt =$$

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} [\cos(2n-1)t + \cos(2n+1)t]dt = \frac{2}{\pi}\left[\frac{(-1)^{n+1}}{2n-1} + \frac{(-1)^n}{2n+1}\right] \tag{2.24}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin 2ntdt = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \sin 2ntdt = 0 \tag{2.25}$$

The last equation is a result of the fact that $f(t)$ is even and $\sin 2nt$ is odd. Consequently, $f(t)\sin 2nt$ is odd, and whenever an odd function is integrated over an interval which is symmetric around zero, the result of the integration is zero. Based on the above equations we can write:

$$f(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty}\left\{\frac{2}{\pi}\left[\frac{(-1)^{n+1}}{2n-1} + \frac{(-1)^{n+1}}{2n+1}\right]\right\}\cos 2nt \tag{2.26}$$

Also, if we write out a couple of terms from the above equation we get:

$$f(t)=\frac{2}{\pi}\left[1+\frac{2}{3}\cos 2t-\frac{2}{15}\cos 4t+\frac{2}{35}\cos 6t+\cdots\right] \tag{2.27}$$

(b) Find the exponential Fourier series of the signal $f(t)$ in Figure 2.6.

We can proceed via two different paths. We can either find, directly, the exponential Fourier series expansion of $f(t)$ by applying the pertinent formulas, or we can use the trigonometric Fourier series expansion already derived to generate the exponential Fourier series expansion. We choose the latter approach, because it is easier. Recall that

$$f(t)=\frac{2}{\pi}+\sum_{n=1}^{\infty}a_n\cos 2nt \tag{2.28}$$

where

$$a_n=\frac{2}{\pi}\left[\frac{(-1)^{n+1}}{2n-1}+\frac{(-1)^{n+1}}{2n+1}\right] \tag{2.29}$$

Due to Euler's identity we can write

$$\cos 2nt=\frac{e^{j2nt}+e^{-j2nt}}{2} \tag{2.30}$$

Using Euler's identity in equation (2.30) we get:

$$f(t)=\frac{2}{\pi}+\sum_{n=1}^{\infty}\frac{a_n}{2}e^{j2nt}+\sum_{n=1}^{\infty}\frac{a_n}{2}e^{-j2nt} \tag{2.31}$$

Notice also that the exponential Fourier series expansion of the signal f(t) has the following form:

$$f(t) = \sum_{-\infty}^{\infty} F_n e^{j2nt} = F_0 + \sum_{n=1}^{\infty} F_n e^{j2nt} + \sum_{n=1}^{\infty} F_{-n} e^{-j2nt} \qquad (2.32)$$

Comparing, one by one, the terms of equations (2.31) and (2.32) we deduce the following identities:

$$F_0 = a_0 \qquad (2.33)$$

$$F_n = \frac{a_n}{2} \qquad (2.34)$$

$$F_{-n} = \frac{a_n}{2} \qquad (2.35)$$

In particular,

$$f(t) = \frac{2}{\pi} + \frac{2}{3\pi} e^{j2t} + \frac{2}{3\pi} e^{-j2t} - \frac{2}{15\pi} e^{j4t} - \frac{2}{15\pi} e^{-j4t} \qquad (2.36)$$

Based on equations (2.33) through (2.35) we can plot the amplitude - frequency, and the phase - frequency plots for the signal $f(t)$. In the case where the coefficients F_n are real we can combine these two plots into one plot; this plot is denoted as the *line spectrum* of $f(t)$. The line spectrum of a signal $f(t)$ corresponds to the plot of F_n's with respect to frequency. The line spectrum of the signal $f(t)$ in this example is depicted in Figure 2.7.

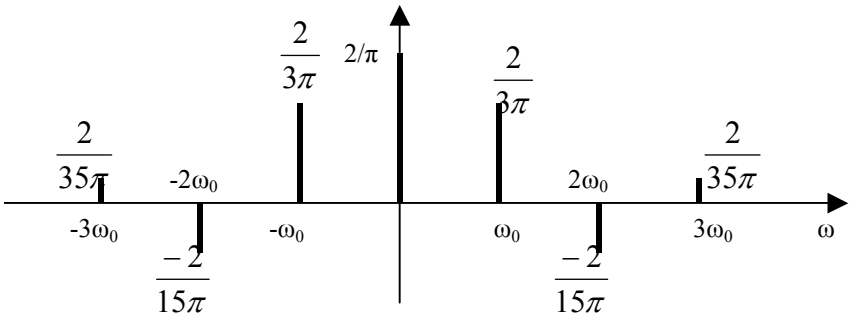


Figure 2.7 Line Spectrum of f(t) of figure 2.6

(c) Plot the right hand sides (RHS's) of equations (2.27) and (2.36)

The plot is shown in Figure 2.8. Note that the RHS's of equation (2.27) and (2.36) is equal to the signal $f(t)$(see Figure 2.6) in the interval $(-\pi/2,\pi/2)$. Also the RHS's of equation (2.27) and (2.36) are periodic with period π**.** Hence the plot of Figure 2.8 is produced by reproducing the plot of Figure 2.7 every π units of time.

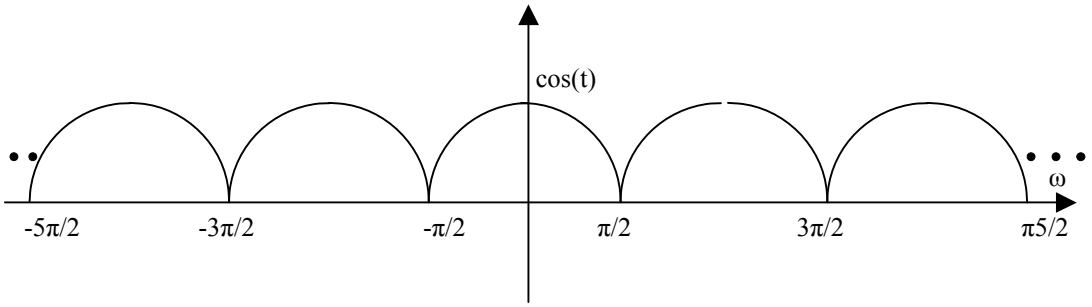


Figure 2.8 Plot of the RHS of equation (2.27)

2.3.2 Example 2

Evaluate the trigonometric Fourier series expansion of the periodic signal $f(t)$ which is depicted in Figure 2.9.

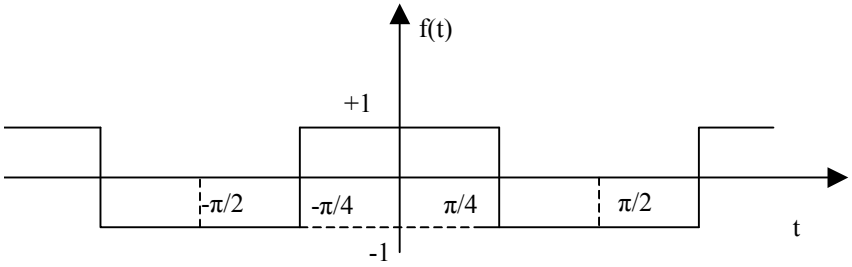


Figure 2.9 plot of f(t)

Let us concentrate on one period of the signal, that is the interval $(-\pi/2,\pi/2)$. Knowing that $T=\pi$ we conclude that $\omega_0 = 2\pi/T = 2$. The trigonometric Fourier series of a signal $f(t)$ is therefore of the form

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2nt + \sum_{n=1}^{\infty} b_n \sin 2nt \qquad (2.37)$$

where

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{\pi}\int_{-T/2}^{T/2} f(t)dt = 0 \tag{2.38}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos 2ntdt$$

$$= \frac{2}{\pi}\int_{-\pi/2}^{-\pi/4}(-\cos 2nt)dt + \frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\cos 2ntdt + \frac{2}{\pi}\int_{\pi/4}^{\pi/2}(-\cos 2nt)dt$$

$$= 2\frac{\sin(n\pi/2)}{n\pi/2} \tag{2.39}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin 2ntdt = 0 \tag{2.40}$$

The last equation is a result of the fact that $f(t)$ is even and $\sin 2nt$ is odd. Consequently, $f(t)\sin 2nt$ is odd, and whenever an odd function is integrated over an interval which is symmetric around zero, the result of the integration is zero. Based on the above equations we can write:

$$f(t) = \sum_{n=1}^{\infty}\left\{2\frac{\sin(n\pi/2)}{n\pi/2}\right\}\cos 2nt \tag{2.41}$$

Also, if we write out a couple of terms from the above equation we get:

$$f(t) = \frac{4}{\pi}\left[\cos 2t - \frac{1}{3}\cos 6t - \frac{1}{5}\cos 10t + \frac{1}{7}\cos 14t + \cdots\right] \tag{2.42}$$

3 Power — Parseval's Relation

3.1 Definitions

The power P_f of a real signal $f(t)$ is defined as follows:

$$P_f = \lim_{T_0\to\infty}\frac{1}{T_0}\int_{-T/2}^{T/2} f^2(t)dt \tag{2.43}$$

It is not difficult to show that if the signal $f(t)$ is periodic with period T, then its power P_f can be computed as follows:

$$P_f = \frac{1}{T}\int_{-T/2}^{T/2} f^2(t)dt \tag{2.44}$$

Parseval's relation for periodic signals says that the power content of a periodic signal is the sum of the power contents of its components in the Fourier series representation of the signal. In particular, if the exponential Fourier series coefficients of $f(t)$ is given by the following equation:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \tag{2.45}$$

then Parseval's theorem says that the power P_f of $f(t)$ is equal to:

$$\sum_{n=-\infty}^{\infty} |F_n|^2 \tag{2.46}.$$

In the above equation $|F_n|^2$ is the magnitude square of the complex number F_n. An alternative expression of Parseval's theorem for periodic signals, says that the power of a periodic signal *f(t)* is equal to:

$$a_0{}^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n{}^2 + b_n{}^2\right) \tag{2.47}$$

where the a_n's and b_n's in the above equation are the trigonometric Fourier series coefficients of the periodic signal $f(t)$.

Besides being an alternative way of calculating the power of a periodic signal, Parseval's relation allows us to calculate the amount of power of a periodic signal that is allocated to each one of its harmonic components.

3.2 Example 3

(a)Find the power contained in the first harmonic of the periodic signal $f(t)$ of Example 1

From the trigonometric Fourier series expansion of $f(t)$ (see equation (2.27)) we observe that the first harmonic of $f(t)$ is equal to:

$$\frac{4}{3\pi}\cos(2t) \tag{2.48}$$

It is not difficult to show that the amount of power contained in the first harmonic is equal to $\frac{[4/(3\pi)]^2}{2}$ as Parseval's relation predicts.

(b) Find the power contained in the first and second harmonics of the periodic signal $f(t)$ of Example 1

From the trigonometric Fourier series expansion of $f(t)$(see equation (2.27), we observe that the first and second harmonic of $f(t)$is equal to:

$$\frac{4}{3\pi}\cos(2t)+\frac{4}{15\pi}\cos(4t) \tag{2.49}$$

It is not difficult to show that the amount of power contained in the first two harmonics is equal to $\frac{[4/(3\pi)]^2}{2}+\frac{[4/(15\pi)]^2}{2}$ as Parseval's relation predicts.

(c) Find the power contained in the DC component of the periodic signal $f(t)$of Example 1

From the trigonometric Fourier series expansion of $f(t)$, (see equation (2.27)) we observe that the DC component of $f(t)$ is equal to $2/\pi$. Hence the amount of power contained in the DC component is equal to $(2/\pi)^2$.

4 Pre-lab Questions

1. Find both the trigonometric and the exponential Fourier series expansion of the signal f(t) in Figure 2.9. Plot the amplitude frequency for the signal. Identify the first three harmonics and their amplitude. Find the power content from the time domain representation of the signal and from the first three harmonics of the Fourier series expansion, and make appropriate remarks.

2. What is the difference between an energy signal and a power signal? Give two examples of each.

5 Implementation

In this lab you will be introduced to the relationship between pulse shapes and their spectra. There are some important concepts to recognize when performing the experiment, such as:

1. When a pulse becomes narrower in the time domain its energy becomes more spread out in frequency.

2. The envelope of the spectrum of a periodic signal is the Fourier transform of the pulse shape.

To illustrate the first concept given above, consider the Fourier Transform of a rectangular pulse shown in Figure 2.11. It can be found from the frequency domain graph (Figure 2.11) that 90% of the energy is concentrated within the main lobe in frequency. As the pulse width is decreased in time, or T decreases, the width of the main lobe in the frequency domain spreads, thus spreading out the energy of the signal.

To illustrate the second concept, recall that when two signals are convolved in the time domain, their transforms are multiplied in the frequency domain.

Any periodic signal can be represented mathematically described as the basic pulse shape convolved with a time sequence of pulses. Thus the spectrum of the periodic signal can be derived as shown in Figure 2.12.

6 Procedure

EQUIPMENT:

TDS5052B Oscilloscope
Function Generator
You should bring a floppy disk to store your waveforms.

6.1. Sine Wave

In this section you will observe when the signal frequency of a sine wave is shifted, the spectral line will also shift in frequency.

a. Turn on the Oscilloscope and allow a minute for the instrument to boot and stabilize. Then, press *Default Setup* to clear settings made by other students. Turn on the Function Generator and connect the output to Channel 1 input of the oscilloscope. Adjust the function generator for the oscilloscope to display 10 cycles of a 1 KHz, 0.5 volt peak sine wave with zero DC offset. Do not change the Horizontal scale setting of the oscilloscope during the remainder of the experiment. Do not change the amplitude or DC offset settings of the Function Generator during the remainder of the experiment.

Note. Be sure the Function Generator Duty Cycle knob is set to Cal.

b. Use the mouse to click the *Math* Button on the Toolbar and select *Spectral Setup* to open the Control Window. Under the *Create* tab, click the *Magnitude* icon and click *Channel 1* as source. Click *Apply.*

c. Click the *Mag* tab and select *Linear* for the vertical scale factor. Click the Scale, set the vertical scale to 100 mv rms per division, and click *Enter*. Click *Apply*. Click *OK* to close the Control Window.

d. Press the *Zoom* icon. Set the Zoom *Position* to 2.5% and the Zoom *Factor* to 20.

e. Determine the amplitude and frequency displayed for the spectrum of the sine wave. Save the waveform to include in your report.

f. Change the Function Generator frequency to 5 KHz. Determine the amplitude and frequency displayed and save the waveform to include in your report.

6.2. Square and Rectangular Waves

A rectangular wave contains many sine wave harmonic frequencies. When the reciprocal of the duty cycle (either the positive or negative duty cycle) is a whole number, the harmonics corresponding to multiples of that whole number will be missing. For example, if the duty cycle is 50%, then 1/0.5 = 2. Thus, the 2nd, 4th, 6th, etc. harmonics will be missing, i.e. zero.
A rectangular wave with a 50% duty cycle is a square wave. For the square wave, the magnitude of the harmonic will be inversely proportional to the harmonic's number. For example, if the magnitude of the 1st harmonic is A, then the magnitude of the 3rd harmonic is A/3, the 5th harmonic's magnitude is A/5, etc.

a. Change the Function Generator frequency to 1 KHz and function to square wave.

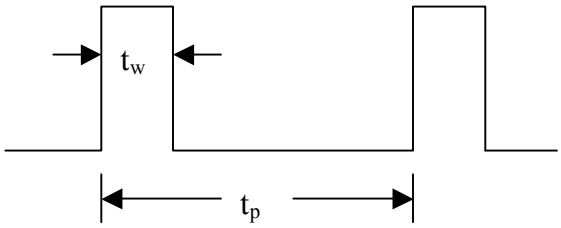
b. Determine the amplitude and frequency displayed for the 1st, 3rd, 5th, and 7th harmonics in the spectrum of the square wave. Identify the zeros of the spectrum envelope (i.e. the frequency locations at which harmonic amplitudes are zero). Save the waveform to include in your report.

c. The duty cycle of a pulse is measured as illustrated in Figure 2-10. The TDS5052B Oscilloscope can automatically measure duty cycle. Click *Measure* on the Toolbar and click *Measurement Setup*. Under Source, click *Channel 1*. Click the *Time* tab and click + *Duty Cyc*. Click *Freq*. Click *Close*. Duty cycle and frequency should now be displayed on the right edge of the screen.

d. Change the duty cycle of the 1 KHz rectangular wave to 10%. Hint: Pull the Duty Cycle Knob to invert the waveform for duty cycles of less than 50%. Changing the duty cycle also changes the frequency, so you will also need to readjust the frequency until you attain both 1 KHz and 10% duty cycle. Notice that a DC component appears in the frequency domain display. The DC component of the signal from the Function Generator is negative. However, the DC component observed in the frequency domain display is positive because the magnitude is the rms value, which is always positive. Determine the amplitude and frequency of the 1st, 2nd, 3rd, 4th, and 5th harmonics. Determine the location of zeros. Save the waveform.

e. Change the duty cycle of the 1 KHz rectangular wave to 25%. Determine the amplitude and frequency of the 1st, 3rd, 5th, 6th, and 7th harmonics. Determine the location of zeros. Save the waveform.

f. Change the duty cycle of the 1 KHz square wave to 90%. Determine the amplitude and frequency of the 1st, 2nd, 3rd, 4th, and 5th harmonics. Determine the location of zeros. Save the waveform.



duty cycle = $(t_w/t_p) * 100\%$

Figure 2.10 Duty cycle of a square wave signal

6.3. Triangular Wave

The triangular wave, like the square wave, only contains odd numbered harmonics. The magnitude of the harmonics is inversely proportional to the square of the harmonic's number. For example, if the 1st harmonic's magnitude is A, then the 3rd harmonic's magnitude will be A/9, the 5th harmonic's magnitude will be A/25, etc.

a. Set the Function Generator Duty Cycle knob to Cal and switch the function to triangular wave. Set the frequency to 1 KHz. Determine the amplitude and frequency of the 1st, 3rd, and 5rd harmonics.

b. Determine the location of zeros. Save the waveform.

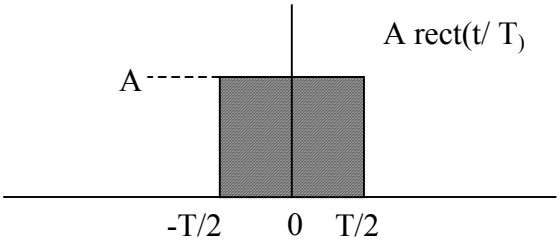
7 Calculations and Questions

1. Compare the waveforms you obtained for rectangular waves on the basis of harmonic location, zero location, and peak spectrum amplitude (i.e. first harmonic level). Explain the results on the basis of the pulse width and signal frequency.

2. Use your data to show that the spectrum of a triangular wave is equal to the spectrum of a square wave squared.

3. The spectrum of the square wave should decay proportional to $1/\omega$ (where $\omega = 2\pi f$). The spectrum of the triangular wave should decay proportional to $(1/\omega)^2$. Verify this using your measurements.

Time Domain



F(A rect(t/T) = ATSin(πfT)/(πfT)

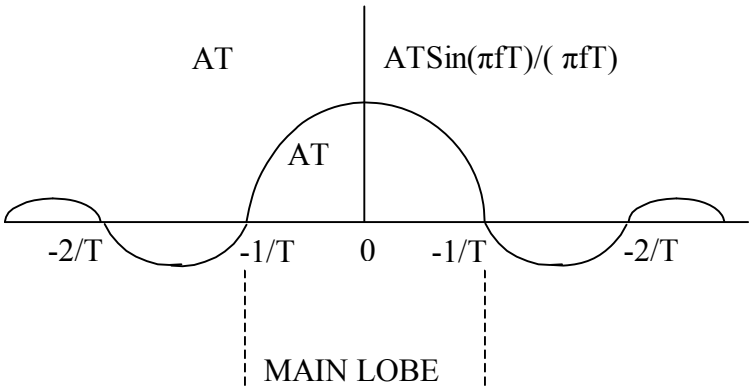


Figure 2.10 Rectangular Function and its frequency representation (Fourier transform)

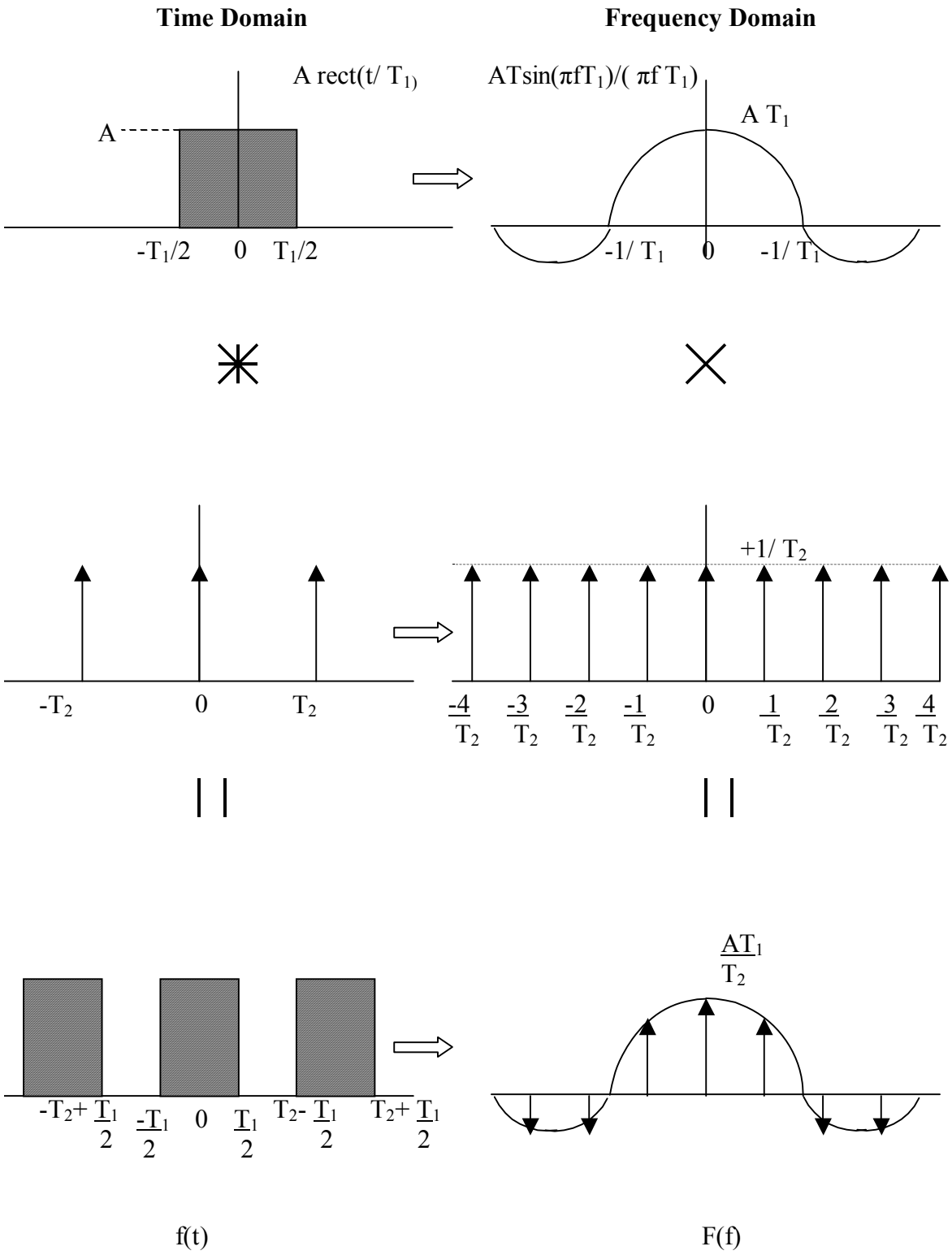


Figure 2.11 Mathematic representation of a periodic signal in the time and frequency domain

EXPERIMENT 3
Low Pass Filter

1 Objective

To observe some applications of low-pass filters and to become more familiar with working in the frequency domain.

2 Theory

2.1 Systems

A communication system consists of three major components: the transmitter, the channel and the receiver. The transmitter and the receiver are comprised of a cascade of black boxes that accept input signals, produce output signals, and they are referred to as systems. This section is devoted to understanding what a system does, and clarifying the various types of systems used in the transmitter and receiver for a communication system.

Definition: A system is a rule for producing an output signal $(g(t))$ due to an input signal $(f(t))$. If we denote the rule as $T[.]$, then

$$g(t) = T[f(t)] \tag{3.1}$$

An electric circuit with some voltage source as the input and some current branch as the output is an example of a system.

Note that for two systems in cascade, the output of the first system forms the input to the second system, thus forming a new overall system. If the rule of the first system is T_1 and the rule of the second system is T_2, then the output of the overall system due to an input $f(t)$ applied to the first system is equal to $g(t)$, such

$$g(t) = T_2[T_1[f(t)]] \tag{3.2}$$

There are a variety of classifications of systems that owe their name to their properties. In this section we will focus only in the classification of systems into the linear versus nonlinear categories and the time-invariant versus time-variant categories.

If a system is linear then the principle of superposition applies. The output of a system with rule $T[.]$ that satisfies the principle of superposition exhibits the following property:

$$T[a_1 f_1(t) + a_2 f_2(t)] = a_1 T[f_1(t)] + a_2 T[f_2(t)] \tag{3.3}$$

Where a_1 and a_2 are arbitrary constants. A system is nonlinear if it is not linear. A linear system (friendly system) is usually described by a linear differential equation of the following form:

$$a_n(t)g^n(t) + a_{n-1}(t)g^{n-1}(t) + \ldots a_1(t)g^1(t) + a_0(t)g^0(t) = f(t) \qquad (3.4)$$

Where $g(t)$ designates the output of the system, while $f(t)$ designates the input of the system. In the above equation $g^k(t)$ denotes the $k-th$ time derivative of the function $g(t)$.

Virtually every system that you consider in your circuit classes (e.g., first order $R-C, R-L$ circuit, second order $R-L-C$ circuits) is an example of linear systems. Any circuit that has components, whose $v-i$ curve is nonlinear (e.g., a diode) is likely to be a nonlinear system.

Another useful classification of systems is into the categories of *Time-Invariant* versus *Time-Varying* systems. A system is time-invariant if a time-shift in the input results in a corresponding time-shift in the output. Quantitatively, a system is time-invariant if

$$g(t-t_0) = T[f(t-t_0)] \qquad (3.5)$$

For any t_0 and any pair of $f(t)$ and $g(t)$, where $f(t)$ denotes an input to the system and $g(t)$, denotes its corresponding output. A system is time-varying if it is not time-invariant.

As we emphasized before, a linear system is described by a linear differential equation of the form shown in equation (3.4). If in the differential equation (3.4) the coefficients $a_k\,(0 \le k \le n)$ are constants and not functions of time, then we are dealing with a linear time-invariant system. Otherwise, we are dealing with a linear time-varying system. Examples of linear time-varying systems from your circuit classes were circuits consisting of R, L and C components and involving one or more switches that were ON or OFF at special instances.

2.2 The Convolution Integral

The convolution integral appears quite often in situations where we deal with a linear time invariant system (LTI). If this system is excited by an input $f(t)$ and the impulse response of the system is $h(t)$, then the output $g(t)$ of the system is equal to the convolution of $f(t)$ and $h(t)$, as illustrated below

$$g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} h(t-\tau)f(\tau)d\tau \qquad (3.6)$$

Note that the impulse response of a linear, time-invariant system is defined to be the output of the system due to an input that is equal to an impulse function, located at $t = 0$. In this manual we are going to demonstrate the convolution of two rectangular pulses.

2.3 Convolution Examples

Consider two rectangular pulses, f_1 and f_2, depicted in Figure 3.1. We are going to find the convolution of these two rectangular pulses. We designate the convolution by $f_3(t)$. That is

$$f_3(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau) d\tau \qquad (3.7)$$

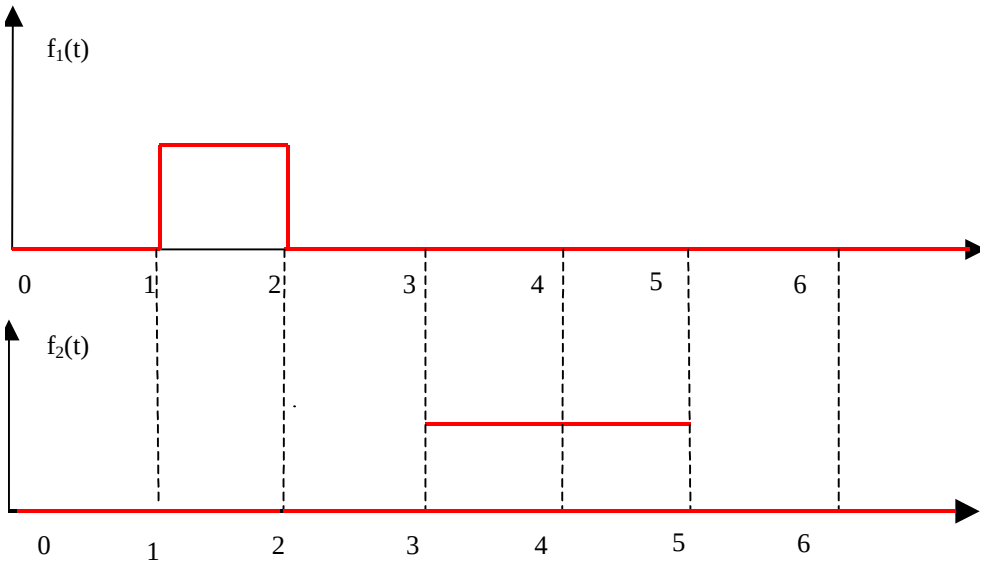


Figure 3.1 two rectangular pulses to be convolved

We want to compute the convolution of $f_1(t)$ and $f_2(t)$. It seems that we have to compute the convolution integral for infinitely many instances of time t (see equation (3.7)). A more careful observation though allows us to distinguish the distinct t-range over which the convolution integral needs to be evaluated. One way of finding the distinct t-range is by remembering that equation (3.7) tells us that we need to slide the rectangle f_1 with respect to the stationary rectangle f_2 and identify the product of these two pulses over the common interval of overlap. It is not difficult to show that the sliding rectangle f_1 (actually $f_1(t-\tau)$) is nonzero over the τ-interval $(t-2, t-1)$. We now distinguish five cases:

Case A: $f_1(t-\tau)$ is to the left of $f_2(\tau)$, and f_1 and f_2 do not overlap (see Figure 3.2a).

In order for Case A to be valid we have to satisfy the constraint that $t-1<3$ or $t<4$. Then is easy to show that

$$f_3(t) = 0 \qquad (3.8)$$

Case B: $f_1(t-\tau)$ is to the left of $f_2(\tau)$, and f_1 overlap f_2 partially from the left (see Figure 3.2b).

In order for Case B to be valid we have to satisfy the constraint that $3 < t-1 < 4$ or $4 < t < 5$. Then is easy to show that

$$f_3(t) = \int_3^{t-1} d\tau = t - 4 \tag{3.9}$$

Case C: $f_1(t-\tau)$ is completely overlapping with (see Figure 3.2c).

In order for Case C to be valid we have to satisfy the constraint that $4 < t-1 < 5$ or $5 < t < 6$. Then is easy to show that

$$f_3(t) = \int_{t-2}^{t-1} d\tau = (t-1) - (t-2) = 1 \tag{3.10}$$

Case D: $f_1(t-\tau)$ is to the right of $f_2(\tau)$, and f_1 overlap f_2 partially from the right (see Figure 3.2d).

In order for Case D to be valid we have to satisfy the constraint that $5 < t-1 < 6$ or $6 < t < 7$. Then is easy to show that

$$f_3(t) = \int_{t-2}^{5} d\tau = 5 - (t-2) = 7 - t \tag{3.11}$$

Case E: $f_1(t-\tau)$ is to the right of $f_2(\tau)$, and f_1 and f_2 do not overlap (see Figure 3.2e).

In order for Case E to be valid we have to satisfy the constraint that $6 < t-1$ or $t > 7$. Then is easy to show that

$$f_3(t) = 0 \tag{3.12}$$

Hence, combining all the previous cases, we get

$$f_3(t) = \begin{cases} 0 & for\ t < 4 \\ t-4 & for\ 4 < t < 5 \\ 1 & for\ 5 < t < 6 \\ 7-t & for\ 6 < t < 7 \\ 0 & for\ t > 7 \end{cases} \tag{3.13}$$

A plot of the function $f_3(t)$ is shown in Figure 3.2f

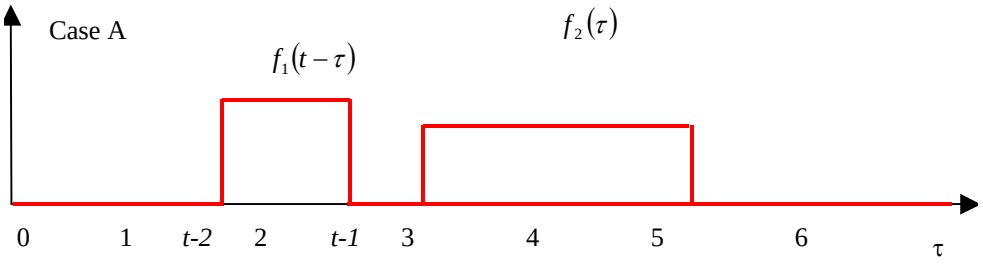


Figure 3.2 a : Convolution under Case A

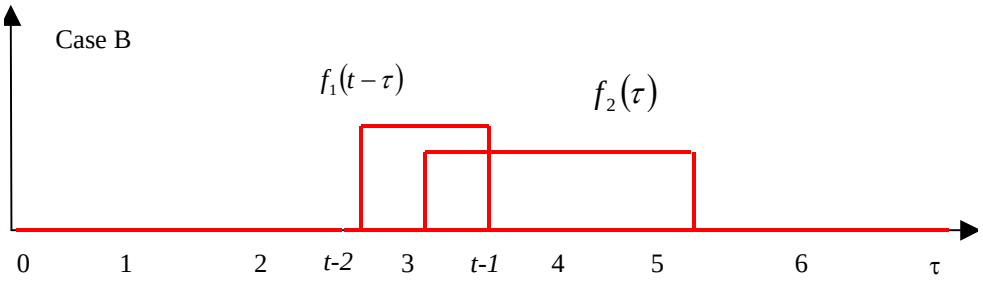


Figure 3.2 b : Convolution under Case B

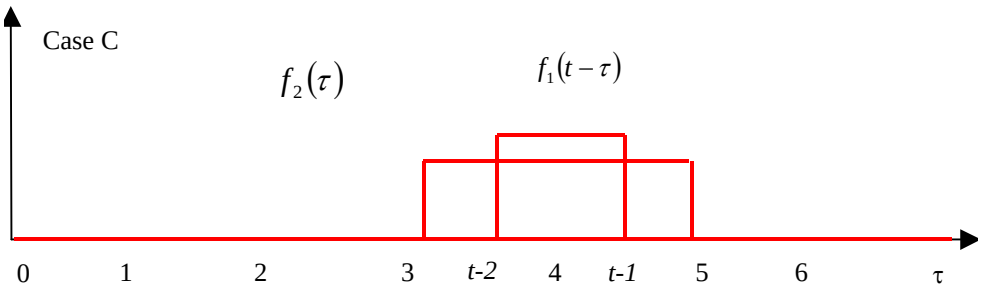


Figure 3.2 c : Convolution under Case C

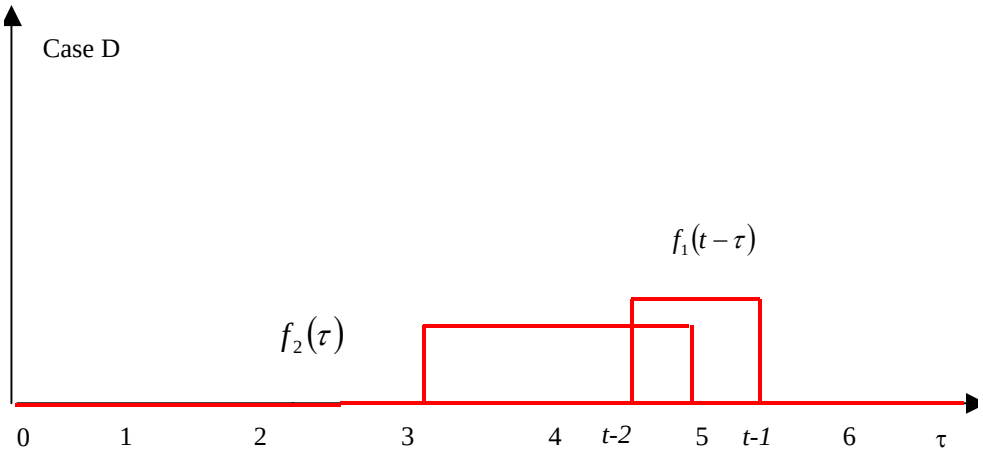


Figure 3.2 d : Convolution under Case D

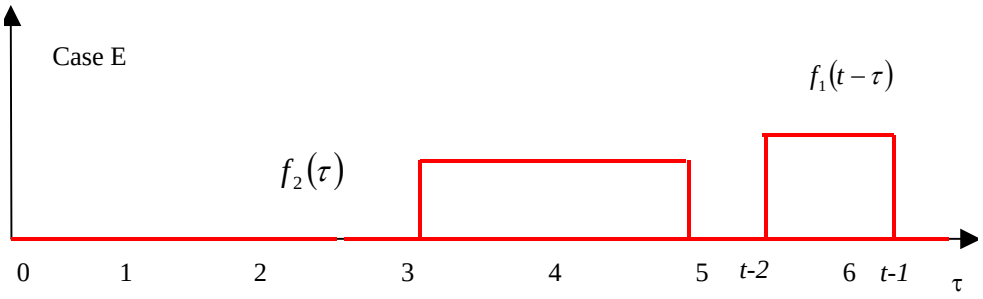


Figure 3.2 e : Convolution under Case E

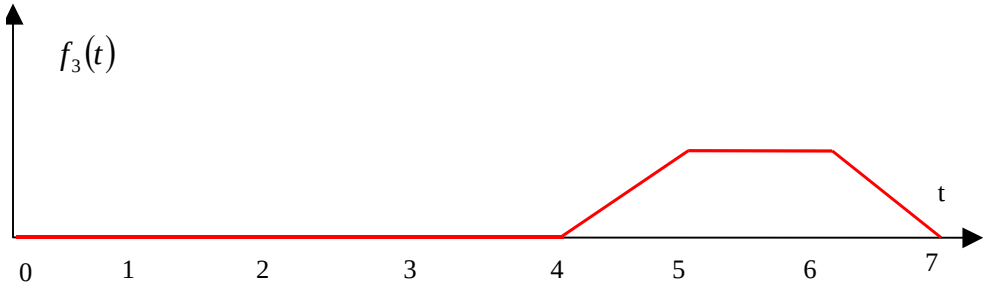


Figure 3.2 f Function $f_3(t)$ (The result of convolution of $f_1(t)$ and $f_2(t)$)

Based on our previous computations we can state certain rules pertaining to the convolution of two rectangles. These rules can be proven following the technique laid out in the previous example. These rules can be used to find the convolution of two rectangle pulses, without actually having to compute the convolution.

2.4 Golden rules for the Convolution of two Rectangle Pulses.

Consider a rectangular pulse $f_1(t)$ with amplitude A_1 over the interval $[x_1, x_2]$, and another rectangular pulse $f_2(t)$ with amplitude A_2 over the interval $[y_1, y_2]$. Denote their convolution by $f_3(t)$. Then,

1. The function $f_3(t)$ is a trapezoid.
2. The starting point of the trapezoid is at position $x_1 + y_1$.
3. The first breakpoint of the trapezoid is
   (a) At position $x_1 + y_2$ if pulse $f_2(t)$ is of smaller width than pulse $f_1(t)$.
   (b) At position $x_2 + y_1$ if pulse $f_1(t)$ is of smaller width than pulse $f_2(t)$.
4. The second breakpoint of the trapezoid is
   (a) At position $x_2 + y_1$ if pulse $f_2(t)$ is of smaller width than pulse $f_1(t)$.
   (b) At position $x_1 + y_2$ if pulse $f_1(t)$ is of smaller width than pulse $f_2(t)$.
5. The end point of the trapezoid is at position $x_2 + y_2$
6. The maximum amplitude of the trapezoid is
   (a) Equal to $A_1 A_2 (x_2 - x_1)$ if the $f_1(t)$ is of smaller width.
   (b) Equal to $A_1 A_2 (y_2 - y_1)$ if the $f_2(t)$ is of smaller width.

A plot of $f_1(t)$, $f_2(t)$ and $f_3(t)$ is shown in Figure 3.3. The technique, described above, to compute the convolution of two rectangular pulses can be generalized, in a trivial way, to compute the convolution of arbitrary shaped, finite-duration pulse.

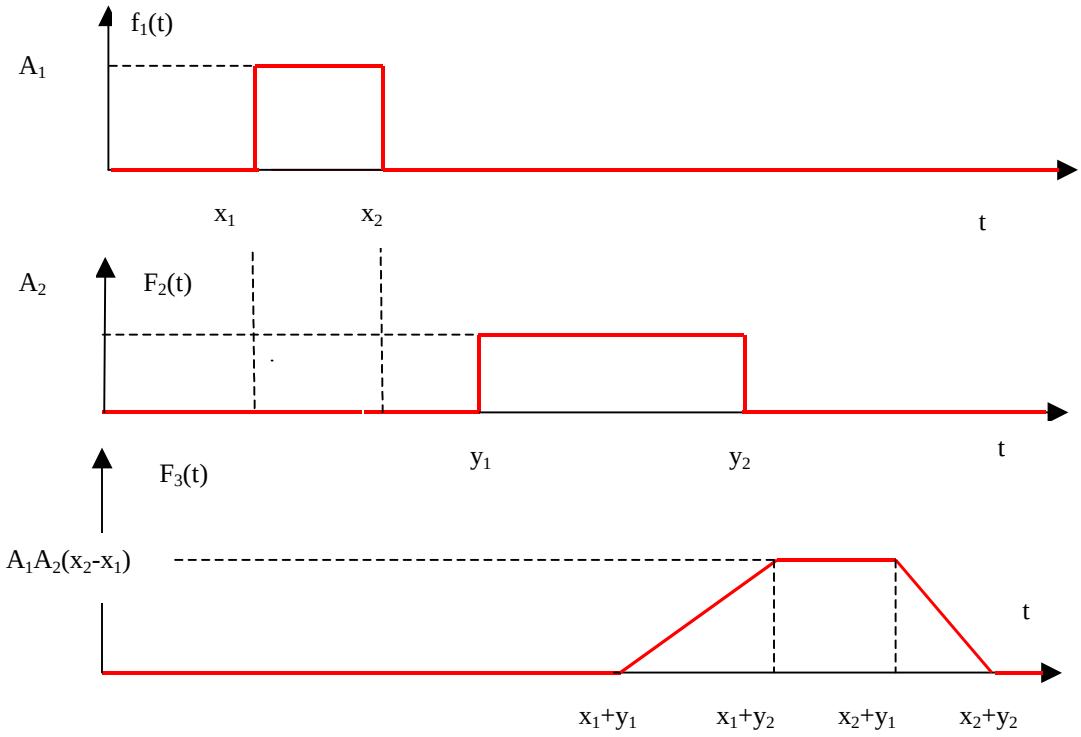


Figure 3.3 Convolution of two Rectangular Pulse

All of the above rules pertaining to the convolution of two rectangular pulses can be proven rigorously. They have been verified above for an example case. In the special case where the pulse (rectangle) $f_1(t)$ and $f_2(t)$ are defined to be nonzero over an interval of the same width convolution $f_3(t)$ turns out to be a triangle. Rules 2 and 5, stated above, for the convolution of two rectangular pulsed extend for the case of the convolution of arbitrary shaped, finite-duration pulses.

Some useful properties of the convolution are listed below. These properties can help us compute the convolution of signals that are more complicated than the rectangular pulses.

1. Commutative Law.

$$f_1(t) * f_2(t) = f_2(t) * f_1(t)$$

2. Distributive Law

$$f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$$

3. Associative Law

$$f_1(t) * [f_2(t) * f_3(t)] = [f_1(t) * f_2(t)] * f_3(t)$$

4. Linearity Law

$$[\alpha f_1(t)] * f_1(t) = \alpha [f_1(t) * f_1(t)]$$

Where α in the above equation is a constant.

2.5 Impulse Function

The impulse function showed up in the discussion of the convolution integral and linear time-invariant system. We said then, that the convolution integral allows us to compute the output of a linear time-invariant system if we know the system's impulse response. The impulse response of a system is defined to be the response of the system due to an input that is the impulse function located at $t = 0$.

The impulse or delta function is a mathematical model for representing physical phenomena that take place in a very small time duration, so small that it is beyond the resolution of the measuring instrument involved, and for all practical purposes, their duration can be assumed to be equal to zero. Examples of such phenomena are a hammer blow, a very narrow voltage or current pulse, etc. In the precise mathematical sense, the impulse signal, denoted by

$$\delta(t)$$

is not a function (signal), it is a *distribution* or a *generalized* function. A distribution is defined in terms of its effect on another function (usually called "test function") under the integral sign.

The impulse distribution (or signal) can be defined by its effect on the "test function" $\phi(t)$, which is assumed to be continuous at the origin, by the following relation:

$$\int_a^b \phi(t)\delta(t)dt = \begin{Bmatrix} \phi(0): & a < 0 < b \\ 0: & otherwise \end{Bmatrix}$$

This property is called the sifting property of the impulse signal. In other words, the effect of the impulse signal on the "test function" $\phi(t)$ under the integral sign is to extract or sift its value at the origin. As it is seen, $\delta(t)$ is defined in terms of its action on $\phi(t)$ and not in terms of its value for different values of t.

One way to visualize the above definition of the impulse function is to think of the impulse function as a function determined via a limiting operation applied on a sequence of well-known

signals. The defining sequence of signals is not unique and many sequences of signals can be used, such as

1. Rectangular Pulse:

$$\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau}\left[U\left(t + \frac{\tau}{2}\right) - U\left(t - \frac{\tau}{2}\right)\right]$$

2. Triangular Pulse:

$$\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau}\left[1 - \frac{|t|}{\tau}\right]$$

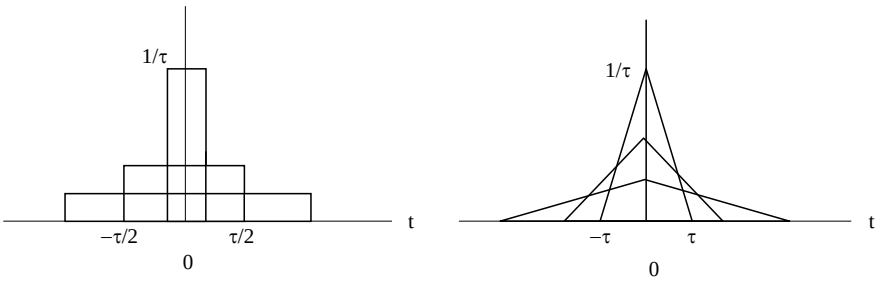
3. Two-sided exponential

$$\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau} e^{-2|t|/\tau}$$

4. Gaussian Pulse

$$\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau} e^{-\pi (t/\tau)^2}$$

The above functions are depicted in Figure 3.4



(a) (b)

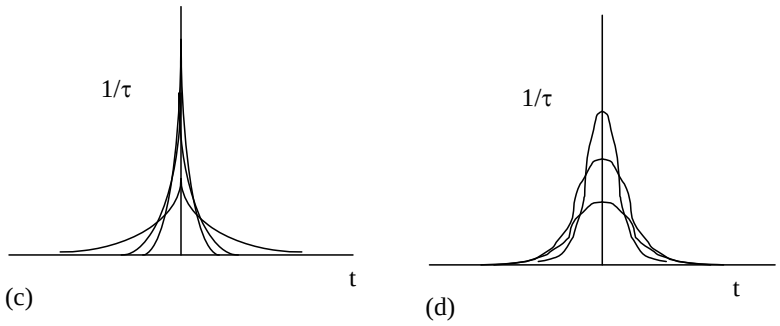


Figure 3.4 Function sequence definition of impulse function:
(a) rectangular pulses; (b) triangular pulses; (c) two-sided exponentials; (d)Gaussian pulses.

The impulse function that we have defined so far was positioned at time $t = 0$. In a similar fashion we can define the shifted version of the impulse function. Hence, the function $\delta(t - t_0)$ designates an impulse function located at position (time) t_0, and it is defined as follows:

$$\int_a^b \phi(t)\delta(t - t_0)dt = \begin{cases} \phi(t_0): & a < t_0 < b \\ 0: & otherwise \end{cases}$$

The impulse function has a number of properties that are very useful in analytical manipulations involving impulse functions: They are listed below:

- Area (Strength): The impulse function $\delta(t)$ has unit area. That is

$$\int_a^b \delta(t - t_0)dt = 1 \quad a < t_0 < b$$

- Amplitude:

$$\delta(t - t_0) = 0 \quad for\ all\ t \neq t_0$$

- Graphic representation

See Figure 3.5

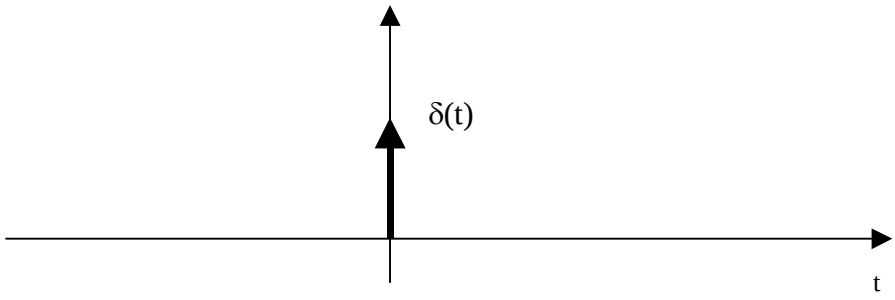


Figure 3.5 Function δ(t)

- Symmetry

$$\delta(t) = \delta(-t)$$

- Time Scaling

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

- Multiplication by a time function

$$f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0)$$

- Relation to the Unit Step Function:

The unit step function is the function defined by

$$U(t - t_0) = \begin{cases} 1: & t > t_0 \\ 0: & t < t_0 \end{cases}$$

It is not difficult to see that

$$\int_{-\infty}^{t} \delta(\tau - t_0)\,d\tau = U(t - t_0)$$

And that

$$\delta(\tau - t_0) = \frac{d}{dt} U(t - t_0)$$

- Convolution Integral.

$$f(t) * \delta(t) = f(t)$$

$$f(t) * \delta(t - t_0) = f(t - t_0)$$

Similar to the definition of $\delta(t)$ we can define $\delta^{(1)}(t)$, $\delta^{(2)}(t)$, …, $\delta^{(n)}(t)$, *the generalized derivatives* of $\delta(t)$ by the following equation:

$$\int_a^b \delta^{(n)}(t)\phi(t)dt = \begin{cases} (-1)^n \dfrac{d^n}{dt^n}\phi(t)\Big|_{t=0} : \quad a < 0 < b \\ 0 : \quad otherwise \end{cases}$$

We can generalize this result to

$$\int_a^b \delta^{(n)}(t - t_0)\phi(t)dt = \begin{cases} (-1)^n \dfrac{d^n}{dt^n}\phi(t)\Big|_{t=t_0} : \quad a < t_0 < b \\ 0 : \quad otherwise \end{cases}$$

For even values of n, $\delta^{(n)}(t)$ is even, and for odd values of n, $\delta^{(n)}(t)$ is odd.

2.6 The Frequency Transfer Function

Consider a linear-time invariant system, which is described by the following differential equation.

$$\sum_{m=0}^{M} a_m \frac{d^m g(t)}{dt^m} = \sum_{k=0}^{K} b_k \frac{d^k f(t)}{dt^k} \tag{3.14}$$

In the above equation, $f(t)$ represents the input to the system, and $g(t)$represents the output of the system. The expression $\dfrac{d^0 f(t)}{dt^0}$ denote 0-th derivative of $f(t)$; same convention holds for $g(t)$.

Assume for a moment that the input $f(t)$ to the above system is equal to $e^{j\omega t}$. Then, we prove that the output of the system is equal to $g(t) = H(w)e^{j\omega t}$ with

$$H(\omega) = \frac{\sum_{k=0}^{K} b_k (j\omega)^k}{\sum_{m=0}^{M} a_m (j\omega)^m} \tag{3.15}$$

In actuality we are contending that if the input to the system is a complex exponential function ($e^{j\omega t}$) the output of the system will be the same exponential function times a constant ($H(\omega)$) which depends on the input parameters (ω) and the system parameters (the a's and b's) when we are referring to $H(\omega)$ as being a constant we mean that it is independent of time. The constant $H(\omega)$ is called the *transfer function of the system.* When the Fourier transform is introduced it will be easy to prove that the transfer function of a system is the Fourier transform of the impulse response of the system. We have already stated that the impulse response of a linear time invariant system describes the system completely. We can make a similar statement about the transfer function of a linear, time-invariant system. That is, if we know the transfer function of a LTI system we can compute the output of this system due to an arbitrary input. Since, we are still operating in the context of Fourier series expansions, and since Fourier series expansions are most appropriate for periodic signals, consider for a moment a periodic input $f(t)$ applied to the system of equation (3.14). Obviously, $f(t)$ has a Fourier series expansion whose form is given below.

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \tag{3.16}$$

It is not difficult to demonstrate that in this case the output $g(t)$ of our system will be equal to

$$g(t) = \sum_{n=-\infty}^{\infty} H(n\omega_0) F_n e^{jn\omega_0 t} \tag{3.17}$$

Equation (3.17) validates out claim that the frequency transfer function of a linear time invariant system is sufficient to describe the output of the system due to an arbitrary input (at least for the case of an input which is periodic). After the Fourier transform is introduced we will be able to extend this result to aperiodic inputs as well. Looking at equation (3.17) more carefully we can make a number of observations:

Observation 1: If the input to a linear time-invariant system is periodic with period $T_0 = \frac{2\pi}{\omega_0}$, then the output of this system is periodic with the same period.

Observation 2: Only the frequency components that are present in the input of a LTI system can be present in the output of the LTI system. This means that a LTI system cannot introduce

new frequency components other than those present in the input. In other words all systems capable of introducing new frequency are either nonlinear systems or time-varying systems or both.

2.7 Frequency transfer Function Example

Let $f(t)$ be a signal as shown in Figure 3.6. This signal is passed through a system (filter) with transfer function as shown in Figure 3.7. Determine the output $g(t)$ of the filter.

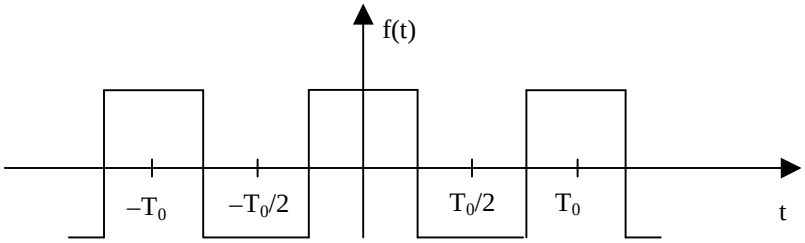


Figure 3.6 *input signal f(t) to the filter in "Frequency Transfer Function Example"*

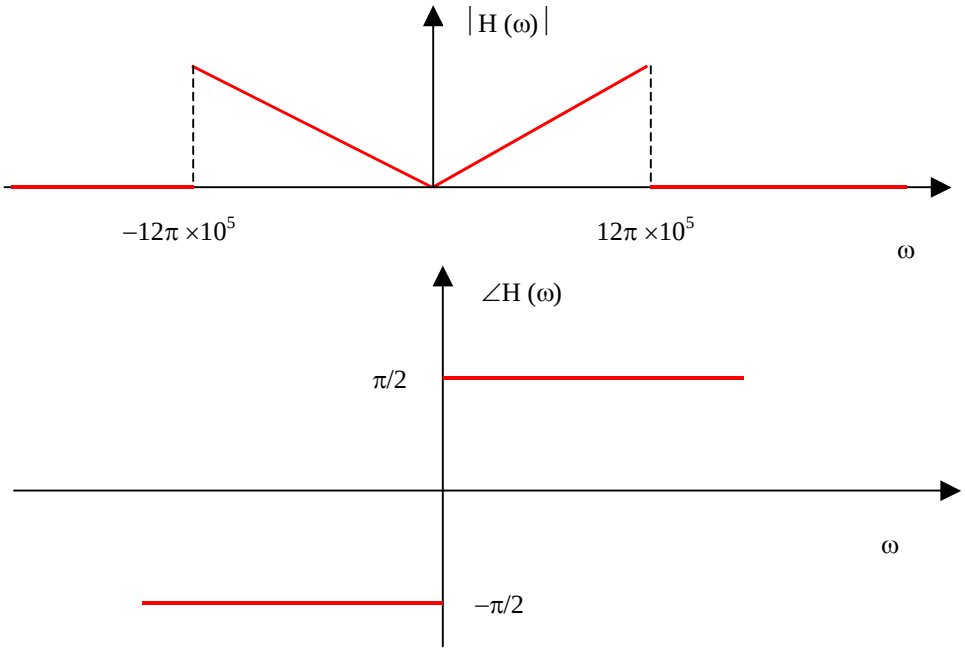


Figure 3.7 *Transfer function of the filter in "Frequency Transfer Function Example"*

We first start with the Fourier series expansion of the input. This can be obtained by following the Fourier series formulas. It can be shown that

$$f(t)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cos\left[2\pi(2n+1)10^5 t\right]=\frac{2}{\pi}e^{j2\pi 10^5 t}+\frac{2}{\pi}e^{-j2\pi 10^5 t}$$

$$-\frac{2}{3\pi}e^{j6\pi 10^5 t}-\frac{2}{3\pi}e^{-j6\pi 10^5 t}+\frac{2}{5\pi}e^{j10\pi 10^5 t}+\frac{2}{5\pi}e^{-j10\pi 10^5 t}+\ldots \qquad (3.18)$$

To find the output of the system we need to find the output of the system due to every complex exponential involved in the expansion of $f(t)$. A closer inspection of $H(\omega)$though tells us that only the complex exponential within the passband of the filter will actually produce an output (the passband is the range of frequencies from $-12\pi 10^5$ rads to $12\pi 10^5$ rads.) Hence, using the already derived formula that gives the output of an LTI system due to a periodic input, we can write:

$$g(t)=\frac{2}{\pi}e^{j(2\pi 10^5 t+\pi/2)}+\frac{2}{\pi}e^{-j(2\pi 10^5 t+\pi/2)}-\frac{2}{\pi}e^{j(6\pi 10^5 t+\pi/2)}$$

$$-\frac{2}{\pi}e^{j(6\pi 10^5 t+\pi/2)}+\frac{2}{\pi}e^{j(10\pi 10^5 t+\pi/2)}\frac{2}{\pi}e^{-j(10\pi 10^5 t+\pi/2)} \qquad (3.19)$$

2.8 Filters

Filtering has a commonly accepted meaning of separation - something is retained and something is rejected. In electrical engineering, we filter signals usually frequencies. A signal may contain single or multiple frequencies. We reject frequency components of a signal by designing a filter that provides attenuation over a specific band of frequencies, and we retain components of a signal through the absence of attenuation or perhaps even gain over a specified band of frequencies. Gain may be defined as how much the input is amplified at the output.

Filters are classified according to the function they have to perform. Over the frequency range we define pass band and stop band. Ideally pass band is defined as the range of frequencies where gain is 1 and attenuation is 0, and stop band is defined, as the range of frequencies where the gain is 0 and the attenuation is infinite.

Filters can be mainly classified as low pass, high pass, band pass and band stop filters.

A low pass filter can be characterized by the property that the pass band extends from frequency $\omega = 0$ to $\omega = \pm\omega_c$, where ω_c is known to be the cutoff frequency. (See Figure 3.8(a))

A high pass filter is the complement of a low pass filter. Here the previous frequency range $\omega = 0$ to $\omega = \omega_c$ is the stop band and the frequency range from $\omega = \omega_c$ to positive infinity and $\omega = -\omega_c$ to negative infinity is the pass band. (See Figure 3.8(b))

A band pass filter is defined as the one where frequencies from ω_1 to ω_2 are passed while all other frequencies are stopped. (See Figure 3.8(c))

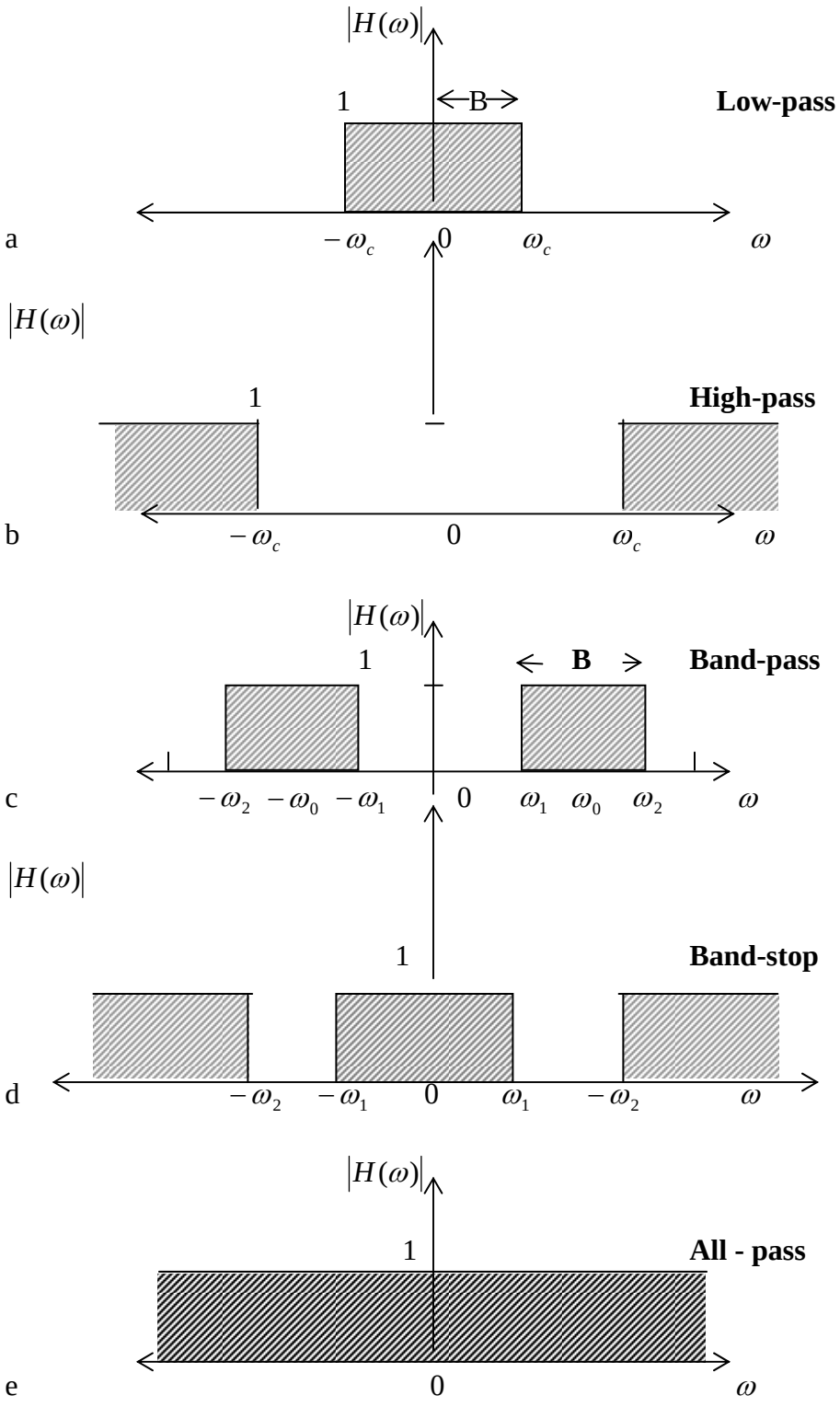


Figure 3.8 The magnitude of the frequency transfer function of low-pass, high-pass, band-pass, band-stop and all-pass. In all cases the responses are ideal

A band stop filter is the compliment of the band pass filter. Frequencies from ω_1 to ω_2 are stopped here and all other frequencies are passed. (See Figure 3.8(d))
In the Figure 3.9 below a practical (non-ideal) 6^{th} order Butterworth low-pass filter is shown.

In Figure 3.9 the various filter characteristics such as pass band, cut-off frequency are clearly indicated. In figure 3.10 we are illustrating how the filter characteristics change with the order of the filter.

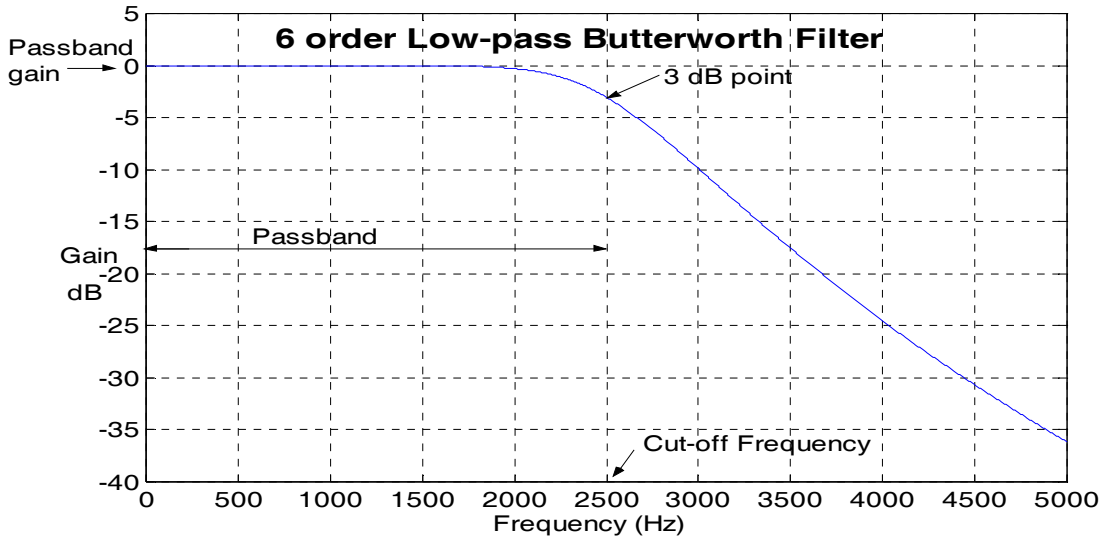


Figure 3.9 6-th order Low-pass Butterworth Filter

The following Figure 3.10 shows how the order of a filter changes the frequency response. The higher the order, the closer the filter behaviors as ideal.

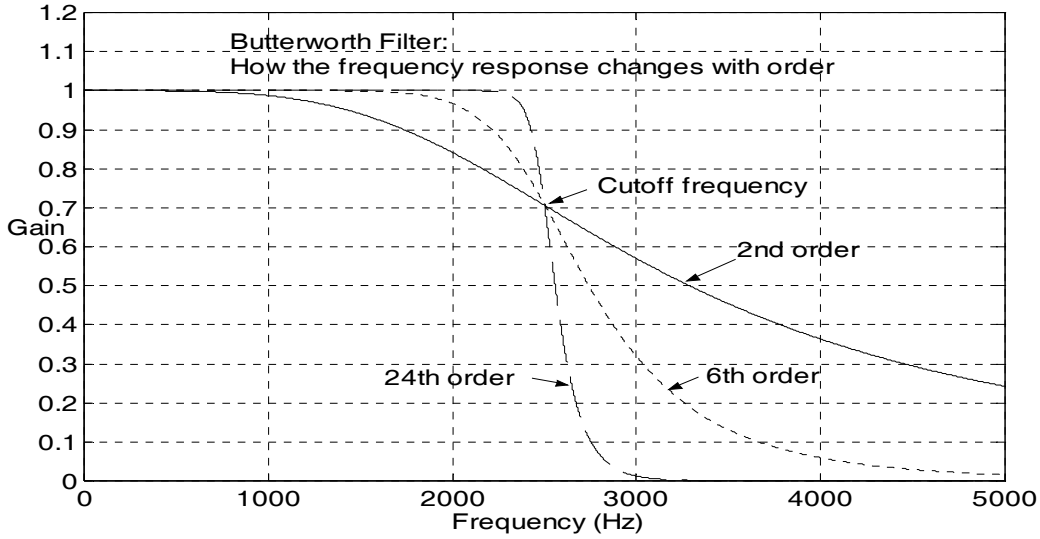


Figure 3.10 Butterworth Filters *(How the frequency response changes with order)*

Filters are key components in communication systems. An ideal filter will only allow a specified set of frequencies to pass from input to output with equal gain. However, no filter is ideal, and as a result there are many types of filters, that are used in modeling an ideal filter closely in one or more aspects. Filters have many applications besides separating a signal form surrounding noise or other signals. Some of these applications are listed below.

1. Integration of a signal
2. Differentiation of a signal
3. Pulse shaping
4. Correcting for spectral distortion of a signal
5. Sample and hold circuits

Two Butterworth low pass filters (LPF's) will be used in this lab: a first order and a fourth-order Butterworth filter. The general transfer function of a Butterworth LPF is:

$$|H(\omega)|^2 = \frac{A}{(1+\omega/\omega_0)^{2n}}$$

Where ω_0 = the filter cutoff frequency and
n = the order of the filter

The actual transfer functions of the filters that are going to be used in this lab are:

1st order LPF: $|H(s)| = \frac{1}{1+RCS}$ (3.20)

4th order LPF:

$$H_L(s) = \frac{\left(1+\frac{R_2}{R_1}\right)\left(1+\frac{R_4}{R_3}\right)}{\left((RCS)^4 + \left(3-\left(\frac{R_1+R_2}{R_1}\right)RCS+1\right)(RCS)^2 + \left(3-\left(\frac{R_1+R_2}{R_1}\right)RCS+1\right)\right)}$$

Where $\omega_0 = 1/RC$ for both filters

Two applications for low pass filters will be illustrated in this lab:

1. A LPF used as an integrator
2. A LPF used as a first harmonic isolator (square-to-sine conversion).

2.8.1 Integrator

The Laplace representation of integration is $1/s$ (where $s = j\omega$). A first order low pass filter can approximate an integrator when $\omega >> \omega_0$ $\{then \quad V_0/V_{in} = 1/(s/\omega_0)\}$

For an n^{th} order low-pass filter, the transfer function will approximate n integrators in series at $\omega >> \omega_0$, or $\dfrac{V_0(s)}{V_{in}(s)} = \dfrac{1}{(s/\omega_0)^n}$

For simplicity, in this lab a first order LPF will be used to perform a single integration on a signal.

2.8.2 Square-to-sine-Converter

Any periodic signal can be expressed as an infinite sum of orthogonal sinusoids. For orthogonality, each sinusoid (or "harmonic") must have a different frequency, and frequency must be an integer multiple of the frequency of the periodic signal.

For example, if the frequency of a square wave is f_s, then the n^{th} harmonic will have a frequency of $n * f_s$ (where n is any integer from one to infinity). You will see that some of these harmonics will have zero or negative amplitude.

The harmonics of a square wave of frequency f_s are shown in Figure 3.11 (Not to scale); the DC component of the square wave is assumed to be equal to zero.

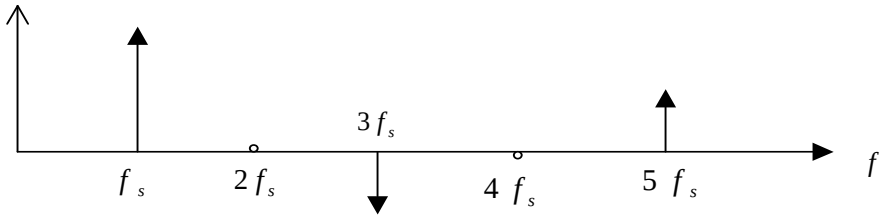


Figure 3.11 The harmonics of a square wave form of frequency f_s

If the square wave excites the input of the low pass filter such that the first harmonic is in the pass band of the filter transfer function and the remaining harmonic are outside of the pass band, then the output of the low pass filter will be a single-tone signal (sinusoid).

3. Simulation

We will use Matlab simulation to see the response of our filter circuits. For simulation you will need the transfer functions of both filter circuits shown in figure 3.15 and 3.16. The transfer functions were given earlier in this manual (See equation 3.20 and 3.21.)

In both filters $R = 1.6k\Omega$ and $C = 0.01\mu F$. For the second filter $R_1 = 10k\Omega$, $R_2 = 12k\Omega$, $R_3 = 10k\Omega$, $R_4 = 1.5k\Omega$ and $C = 0.01\mu F$. With the above values for resistors and capacitors the transfer functions will have the following form (notice that the coefficients are in descending order of s).

$$H_1 = \frac{1}{1+(RC)s}$$

$$H_2 = \frac{2.2\times 1.5}{(RC)^4 s^4 + 2.65(RC)^3 s^3 + 3.48(RC)^2 s^2 + 2.65RCs + 1}$$

Construct a frequency array as following.

w=[0:100:100000];

Construct the coefficient vectors as follows:

b=[coefficients of numerator separated by comma];
a=[coefficients of denominator separated by comma];

Use the MATLAB ‘freqs’ function to generate frequency response of the corresponding filter.

Plot the frequency response of both filters in a single graph using the following a Matlab function.

Note: The freqs function is not available in all MatLab installations. It is installed in all EE Labs at UCF.

Apply the above procedure for both the filters. Note that the frequency response you obtained from the MATLAB simulation that is gain versus angular frequency in radian. You can also plot gain versus frequency (Hz). In that case you have to use the $f=w/2\pi$ vector as the parameter in plot function.

The simulated gain versus frequency (Hz) plot for both filters (2^{nd} order and 6^{th} order) is shown below as Figure 3.12.

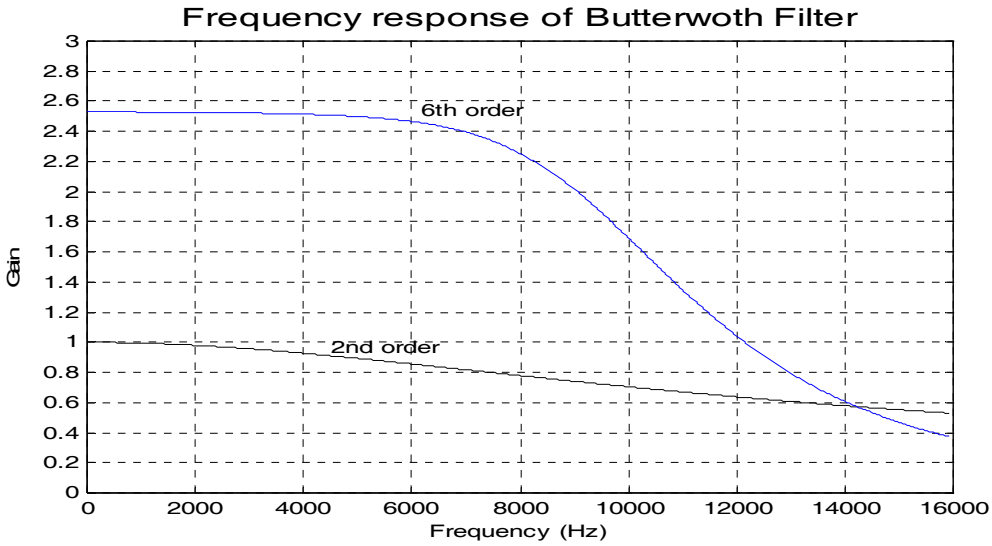


Figure3.12 The simulated gain versus frequency (Hz) plot for both filters

(2^{nd} order and 4^{th} order)

Compare the response of two filters. After the hardware experiment compare the experimental results with the simulated results.

4. Prelab Questions:

1. Calculate the Trigonometric Fourier Series of a square wave with the parameters shown (assume it is periodic with periodic T):

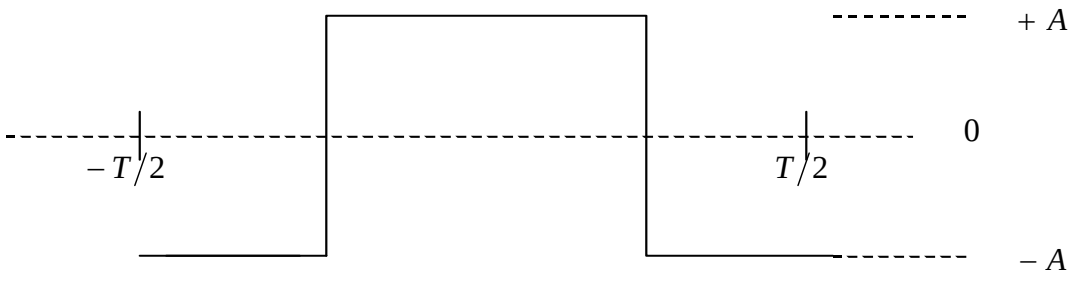


Figure 3.13 a periodic square wave with period T

You may find it easier to add a DC level of A volts. This will only change your spectrum at f=0, where you can subtract the DC back out.

2. Assuming an ideal filter with a cutoff frequency of ω_c ($\omega_c = 2\pi/T$), show graphically, in the frequency domain, why the first harmonic of the square wave should be the only harmonic at the output.

3. Suppose you have an input signal with frequency components from 0 to 2 KHz. Is it possible to design a filter, which will produce an output with frequency components 4 to 6 KHz? Why or why not?

4. What is the difference between dB/octave and dB/decade?

5. Component

In this lab, we will use OpAmp module TL084 which is shown in figure 3.14

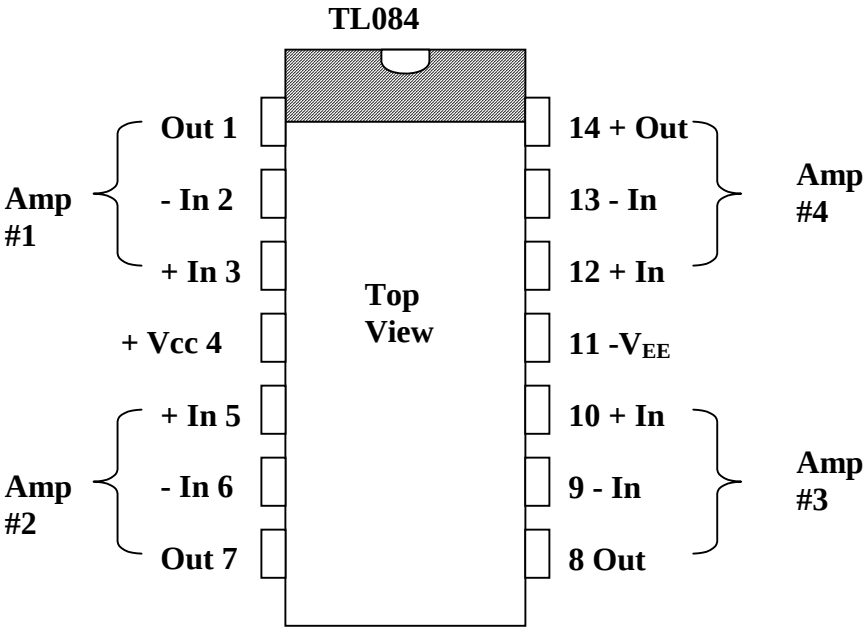


Figure 3.14 Pin layout of OpAmp module TL084

6. Circuit Diagrams:

The circuit diagram of the first order and fourth order low-pass filter used in this lab are shown in Figure 3.15 and Figure 3.16 respectively.

7. Procedure

7.1 Filters

a) Build the lowpass filter shown in Figure 3.15.

b) Using either the oscilloscope, AC voltmeter or spectrum analyzer, determine the cutoff frequency of the filter (-3 dB from the passband), the passband gain, and the filter rate of rolloff (in dB/decade or dB/octave).

c) Build the lowpass filter shown in Figure 3.16 and repeat step (b). **Keep both circuit built for the remainder of the lab.**

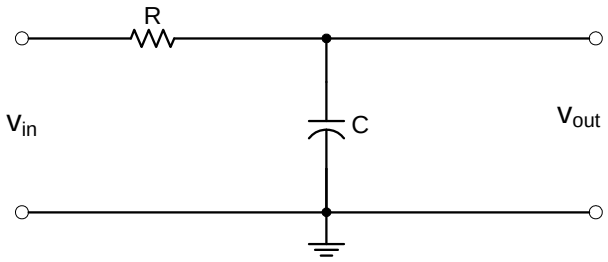


Figure 3.15 *First order low-pass filter*

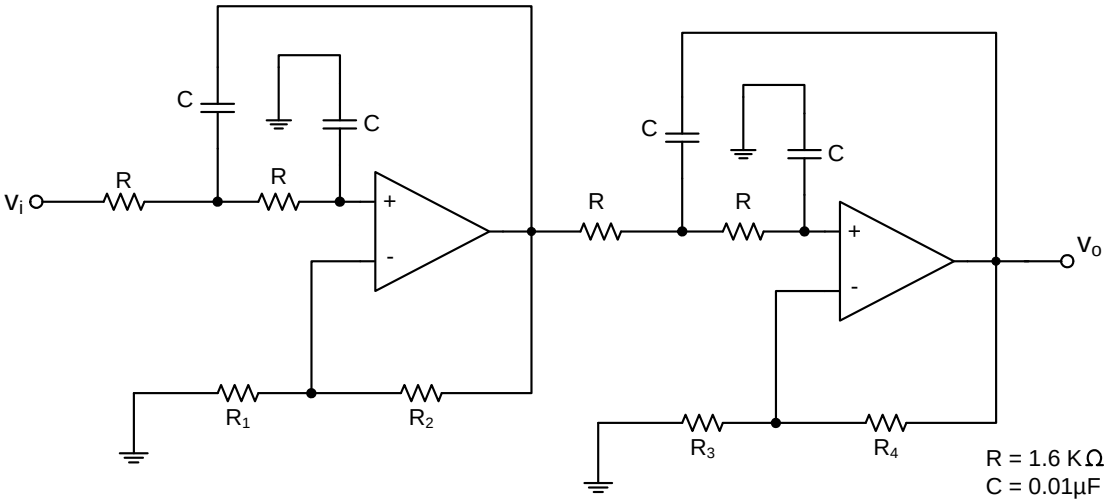


Figure 3.16 *4th-order low-pass filter*

Calculating Filter Parameters

The following procedures will help you understand how to measure filter parameters on the scope or AC voltmeter:

7.2 Cutoff frequency

a) From the function generator, obtain a **.**2 volt, low frequency (10 or 100 Hz range) sine wave. It is important to use a sine wave because there is only one harmonic in a sinusoid, and you only want to test the transfer function at one point in frequency at time. Connect the function generator output to the filter input.

b) Connect the filter output to the oscilloscope (See Figure 3.18) or AC voltmeter input (See Figure 3.17).

c) Vary the input frequency by a factor of 2 or 3 either way and make sure the output level doesn't vary significantly. If it doesn't, then your input frequency. is within the pass band. When you know you are within the pass band, measure the output voltage level (or dB level). This will be your reference level.

d) Vary the input frequency until the output level drops 3 dB below the reference level. On the AC voltmeter this can be read directly. On the oscilloscope, -3 dB is equivalent to the voltage level reduced by a factor of .707 (or, at –3 dB point, Vout(-3dB) = .707 (Vout(ref)). Note this frequency. This is your cutoff frequency.

7.3 Passband gain

a) Using the function generator, generate a .2 volt sine wave at a frequency that you know to be within the filter passband. Connect the function generator output to the filter input.

b) Measure the output voltage (V_0). The filter passband gain is equal to $V_0/V_{in} = V_0/.2$ (See Figure 3.18).

7.4 Rate of rolloff

a) Use the function generator to generate a sinusoid at a frequency that is higher than the cutoff frequency of the filter (maybe by a factor of 2 or so). Record the input amplitude (V_{in}) or measure in dB on the AC voltmeter (you may want to use an input amplitude of 1 volt or more since the output will be attenuated).

b) Connect the function generator output to the filter input. Measure the filter output in volts (V_o) or dB. Calculate V_o/V_{in} and convert to dB ($20\log(V_o/V_{in})$ or subtract the input dB level from the output dB level. This will give you the attenuation, in dB, at your selected frequency.

c) Repeat steps 1 and 2 with a frequency that is a factor of 2 higher than the first frequency used. If you take the difference between the two calculated values of attenuation, in dB, this will be your rate of rolloff in dB/octave. If you want to convert, (N) dB/octave corresponds to ((N) * (20/6)) dB/decade.

7.5 **Square Wave Spectrum**

a) Use the function generator to generate a square wave of **0.2** volts amplitude and **1kHz** frequency, and connect its output to Cannel 1 input of the oscilloscope.

b) Set up the display of the Oscilloscope in order to see 10cycles of the square wave.

c) Generate a spectrum analysis waveform in dBV with a reference voltage offset of 1Volt.
- From the Math Menu, select Spectral Set Up. Create the Magnitude spectra by clicking in the *Magnitude* and *channel-1* icons respectively.
- Click Mag Tab and click the dB icon for vertical scale factor.
- At the reference menu:
  Set the Level to 0dB (This will place the M1 label at the upper side of the screen.)
  Set the Level offset to 1 volt (Reference voltage for dBV scale.)
  Set the scale to 10 dB (10dB per division in the vertical scale.)

d) Clear the time domain waveform turning OFF Channel 1.

e) Using Zoom option from the Multiview section. Adjust the Multipurpose Knobs to Factor of 10 (narrows the width of the zoom window to 12.5KHz) and to a Position of 5% (place the left side limit of the zoom window to 0dB.) You may need to press the HORIZ bottom to re-adjust your measurements.

You should see the first six nonzero harmonics of the square wave displayed. Record the amplitude (in dBV) and frequency of each harmonic. You can use the Cursor in cursor window type.

7.6 Integrator

a) Generate a square wave of amplitude 5V (peak) and frequency= $10 \times \omega_c$ (ten times the cutoff frequency). Apply this signal to the input of the filter from Figure 3.15 (the first order filter)

b) Sketch the output of the filter from the oscilloscope, recording signal peak-to-peak amplitude.

7.7 Square-to-Sine conversion (First Harmonic Isolation)

a) Obtain a .2 volt peak square wave with frequency $f = .75 \times f_c$ from the function generator.

b) Apply the signal from (a) to the input of the filter in Figure 3.15. Sketch the output signal in both the time and frequency domains, scaling axes.

c) Repeat b) with the filter in Figure 3.16

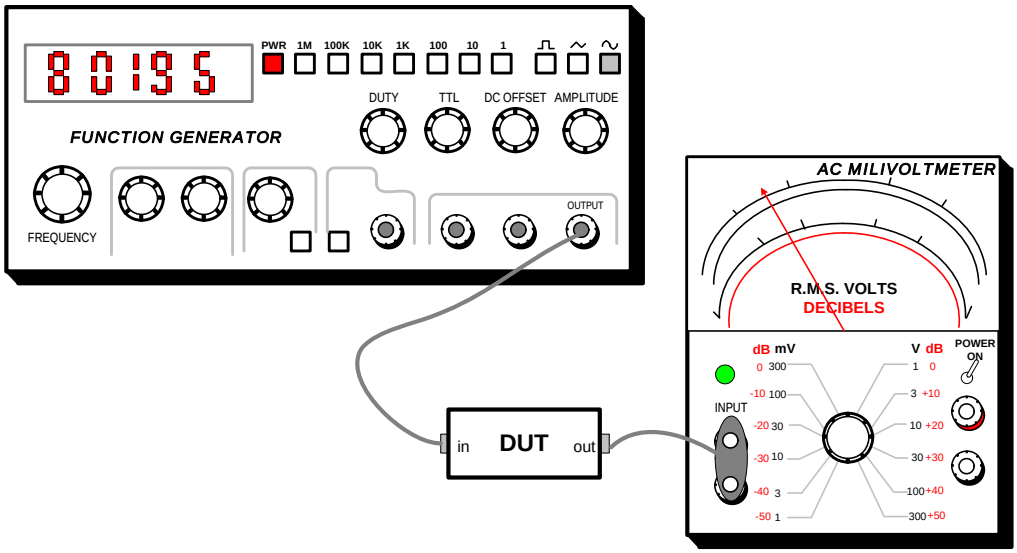


Figure 3.17 Measuring cutoff frequency using AC Voltmeter

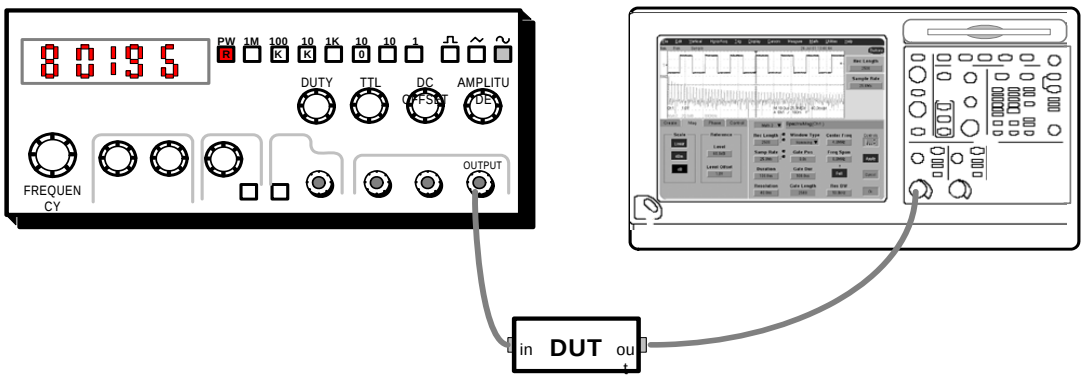


Figure 3.18 Measuring passband using the oscilloscope.

8 Calculations and Questions

1. Convert the values you measured from the spectrum analyzer in dBV to volts and compare to your Fourier Series Coefficients from (1). Recall that the measured amplitudes are RMS amplitudes, so the values must be multiplied by the square root of 2.

2. Compare your results from section IV parts (b) and (c) on the basis of filter gain and filter roll off rate that you measured.

3. Using the waveform shown in question (1), Integrate (in time) to show that the integral of a square wave is a triangular wave. What peak amplitude do you calculate? Recall that you have to solve an indefinite integral to get an answer in functional form.

4. When the first harmonic of the input signal is well beyond the filter cutoff frequency, a first order low pass filter cutoff frequency, a first order low pass filter approximates an integrator (shown in 'Integrator' section of lab introduction). Show that this transfer function, when applied to your square wave, will closely predict the output amplitude you measured. Note that this transfer function can be reduced to a constant multiplied by the integral of the input signal. Also note that your DC level may vary from that calculated, because the DC level will depend on at what point on the square wave the integration began.

Laboratory 4
Amplitude Modulation

1 Objective

To understand the amplitude modulation.

2 Theory

2.1 From Fourier series to the Fourier Integral

The Fourier series is a means for expanding a periodic signal in terms of complex exponentials. This expansion considerably decreases the complexity of the description of the signal, and simultaneously, this expansion is particularly useful when we analyze LTI systems.

We can extend the idea of the Fourier series representation from periodic signals to the case of nonperiodic signals. That is the expansion of a nonperiodic signal in terms of complex exponentials is still possible. However, the resulting spectrum is not discrete any more. In other words, the spectrum of nonperiodic signals covers a continuous range of frequencies. To be able to demonstrate our point, consider for a moment the periodic signal shown in Figure 4.1.

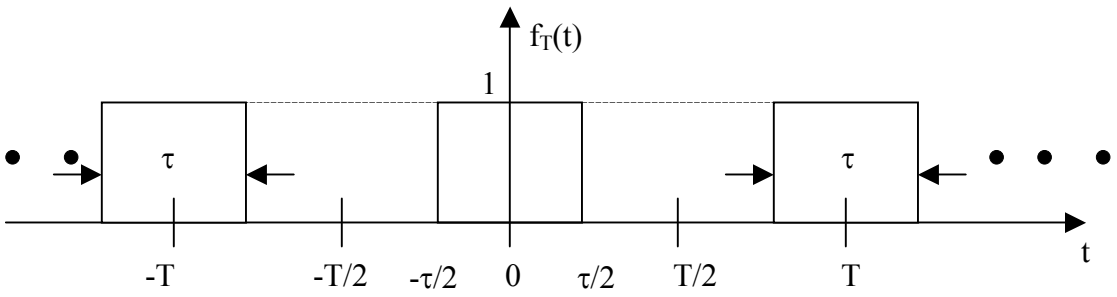


Figure 4.1 A periodic signal $f_T(t)$

We call this periodic signal $f_T(t)$ to remind ourselves that we are dealing with a periodic signal of period T. We can write:

$$f_T(t) = \begin{cases} f(t): & \frac{-T}{2} < t < \frac{T}{2} \\ 0: & otherwise \end{cases} \tag{4.1}$$

where $f(t)$ is equal to the periodic signal $f_T(t)$ over the interval $(-T/2,T/2)$ and zero otherwise. Obviously, $f_T(t)$ has a Fourier series expansion, such that

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \tag{4.2}$$

With

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t} dt = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt \tag{4.3}$$

Also, since $f_T(t)$ and $f(t)$ are equal over the interval $(-T/2,T/2)$ we can write that

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{1}{T} F(\omega_n) e^{j\omega_n t} \tag{4.4}$$

where

$$F(\omega_n) = TF_n \tag{4.5}$$

and

$$\omega_n = n\omega_0 \tag{4.6}$$

Let us now define

$$\Delta\omega = \omega_{n+1} - \omega_n = \omega_0 = \frac{2\pi}{T} \tag{4.7}$$

Then, from equation (4.7) we get

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} F(\omega_n) e^{j\omega_n t} \tag{4.8}$$

Let now T go to ∞, or equivalently, $\Delta\omega$ goes to zero. It is easy to see that the equation (4.8) becomes:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \tag{4.9}$$

Also, it is not difficult to show that

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \tag{4.10}$$

Equations (4.9) and (4.10) are referred to as the *Fourier Transform pair,* that allows us to transform the aperiodic time signal $f(t)$ into the frequency signal $F(\omega)$, and vice versa. Based

on the aforementioned discussion we can now formally introduce the Fourier transform theorem.

2.2 Fourier Transform Theorem

If the signal $f(t)$, satisfies certain conditions known as the Dirichlet conditions, namely,

1. $f(t)$ *is absolutely integrable on the real line, that is,*

$$\int_{-\infty}^{\infty} |f(t)|\,dt < \infty$$

2. *The number of maxima and minima of* $f(t)$ *in any finite interval of the real line is finite.*

3. *The number of discontinuities of* $f(t)$ *in any finite interval of the real line is finite.*

then, the Fourier transform (FT) of $f(t)$, defined by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\,dt \tag{4.11}$$

exists and the original signal can be obtained from its FT by following the rule

$$f_{\pm}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\,d\omega \tag{4.12}$$

where

$$f_{\pm}(t) = \begin{cases} f(t): & \text{if } f(t) \text{ is continuous at } t \\ \dfrac{f(t^{+}) + f(t^{-})}{2}: & \text{if } f(t) \text{ is continuous at } t \end{cases} \tag{4.13}$$

Observation 1:

The Fourier transform, *F(ω)* of a signal, *f(t),* is, in general, a complex function. Its magnitude $|F(\omega)|$ and phase represent the amplitude and phase of various frequency components of $f(t)$.

Observation 2:

If the independent variable in the FT of a signal is chosen to be the frequency f instead of the angular frequency ω we have the following Fourier transform pair:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} < \infty \tag{4.14}$$

and

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \tag{4.15}$$

2.3 Fourier Series to Fourier Transform Example

Let $f_{T_0}(t)$ be a periodic signal depicted in Figure 4.2.

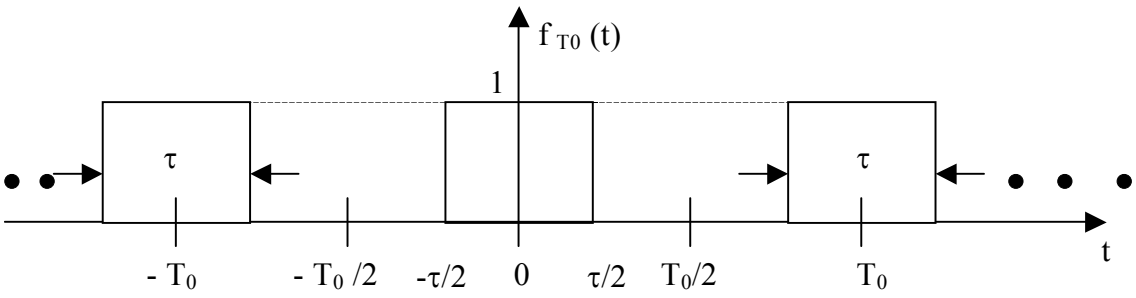


Figure 4.2 A periodic signal f_{T0} (t)

The Fourier series expansion of this signal, of period T_0, is given by the following equation.

$$f_{T_0}(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \tag{4.16}$$

where

$$F_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t) e^{-jn\omega_0 t} dt \tag{4.17}$$

and

$$\omega_0 = \frac{2\pi}{T_0} \tag{4.18}$$

We can compute F_n's as follows:

$$\begin{aligned} F_n &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\, e^{-jn2\pi t/T_0}\, dt = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} f(t)\, e^{-jn2\pi t/T_0}\, dt \\ &= \frac{1}{T_0}\frac{T_0}{-jn2\pi}\left[e^{-jn\pi\tau/T_0} - e^{+jn\pi\tau/T_0}\right] = \frac{\sin(n\pi\tau/T_0)}{n\pi\tau/T_0}\cdot\frac{\tau}{T_0} \end{aligned} \tag{4.19}$$

The line spectrum of the signal $f_{T_0}(t)$ is shown in Figure 4.3 for $\tau/T_0 = k^{-1}$. Note that the separation of two consecutive lines in the line spectrum of $f(t)$ is equal to $2\pi/T_0$. Hence, as T_0 gets larger and larger (i.e., as $f_{T_0}(t)$ becomes less and less periodic-looking) the line spectrum of $f_{T_0}(t)$ becomes denser and denser. Also, note that the envelope of line spectrum of $f_{T_0}(t)$ is the Fourier transform (within a constant of proportionality) of the aperiodic pulse of width τ, centered at $t = 0$. It can be shown that most of the power of $f_{T_0}(t)$ is located within the frequency interval $[-2\pi/\tau, -2\pi/\tau]$. This indicates to us that as we make the signal faster in the time domain (by decreasing the duration τ of the pulses) the frequency content of the signal gets expanded in the frequency domain. This statement is a manifestation of what is called in communication theory the "time-bandwidth" product. It can be shown that the time-bandwidth" product is constant; that is if we expand the signal in the time domain, we shrink the frequency content of the signal in the frequency domain, and vice versa.

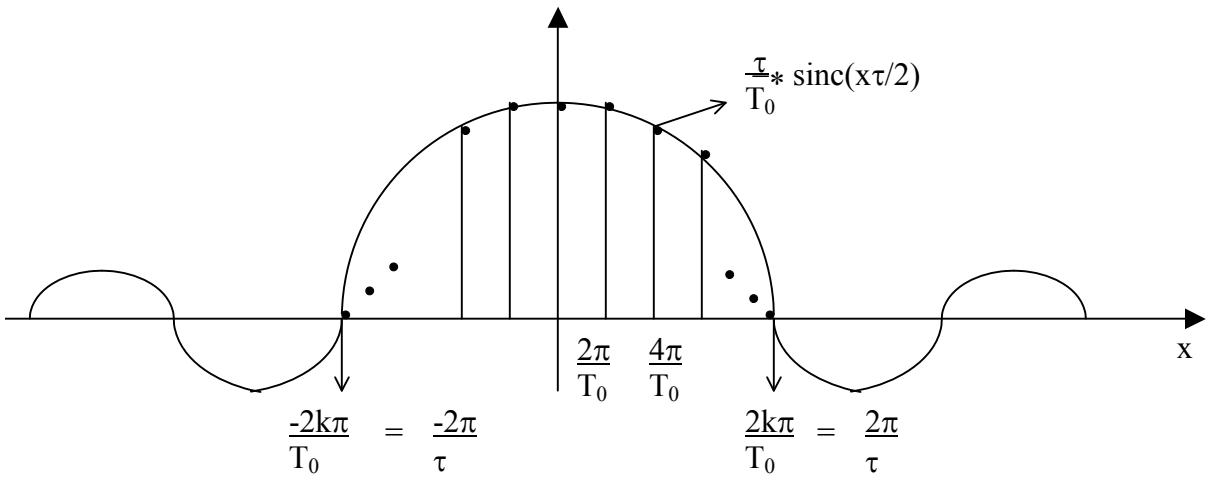


Figure 4.3. The line spectrum of the signal $f_{T0}(t)$

2.4. Fourier Transform Properties

The Fourier transform of an aperiodic signal is as useful as the Fourier series of a periodic signal. As a reminder, the Fourier series of a periodic signal allows us to compute the output of a LTI system due to a periodic input, without having to compute the convolution. Furthermore, the Fourier series of a periodic signal provides us with information pertaining to the frequency content of the signal. Similarly, the Fourier transform of a signal allows to compute the output of a LTI system due to an input, without having to compute the convolution. Also, the Fourier transform of the signal contains information about the frequency content of the signal. Equations (4.11) and (4.12) or equations (4.14) and (4.15) are rarely used for the computation of the Fourier transform of signals. Normally, the Fourier transform of a signal is computed by utilizing Fourier transforms of well known signals (see Table 4.1) and *Fourier Transform* Properties (see Table 4.2).

The *Linearity Property* says that the Fourier Transform (FT) is a linear operation and if We know the FT of two signals We can easily compute the FT of any linear combination of these two signals. The *Scaling Property* says that by contracting (expanding) the signal in the time domain (through multiplication with an appropriate time constant) results in the expansion (contraction) of the FT of the signal (through multiplication with the inverse of the constant); this is another ramification of the "time-bandwidth" product principle. Note that contraction (expansion) of the signal in the time domain corresponds to making the signal faster (slower). The *Delay Property* illustrates something that you might have suspected all along. Delaying a signal in the time domain does not change the frequency content of the signal. The frequency content of the signal is determined by looking at the magnitude of the FT of the signal. The delay property reiterates that a time shift will produce a phase shift in the FT of the signal but it will leave the magnitude of the FT unaltered. The *Convolution Property* assures us that we do not need to perform convolution to produce the output of a LTI system due to an arbitrary input; simply multiplying the FT of the input and the FT of the impulse response of the system allows us to compute the FT of the output of the system. The *Time Differentiation* and *Time Integration* properties demonstrate that making the signal faster/slower (differentiate/integrate) in the time domain results in an expanded/contracted frequency spectrum (FT). The *Duality Property* provides us with a new FT pair, every time a FT pair is computed. In particular, if *F* is the FT of *f,* the duality property says that *F* would a FT that is proportional to the mirror image of *f* with respect to the $\omega = 0$ axis. Finally, the *Amplitude Modulation* property tells us that we can shift the frequency content of a signal at an arbitrary frequency by multiplying the signal with a sinusoid. AM systems are based on this property of the FT.

2.5 Introduction to Communication Systems

The primary function of most communication systems is to transmit information from point *A* to point *B.* Consider, for example, the situation where we have to transmit human voice from point *A* to point *B* and the channel between points *A* and *B* is the air. The human voice at point *A* is first converted, via a microphone, to an electrical signal. Then, we have to transform this

electrical signal into an electromagnetic wave and send it to destination *B.* The way to achieve that is by using an antenna. Knowing though that the maximum frequency content of a voice signal is around 15 KHz we come to the conclusion that the antenna required must be many miles long! A reasonable size antenna could adapt the signal to the channel only if the signal's lower frequency is higher than 800 KHz. Hence, in our case, where a voice signal needs to be transmitted using a reasonable-size antenna it suffices to raise the frequency content of the voice signal to the vicinity of 800 KHz. To accomplish the aforementioned goal we ought to multiply the voice signal with a sinusoidal signal. The result of this multiplication is to latch our $f(t)$ (the voice signal) on the amplitude of the sinusoid, or to modulate the original amplitude of the sinusoid with $f(t)$. This procedure gives rise to a class of systems that are called *amplitude modulation systems.*

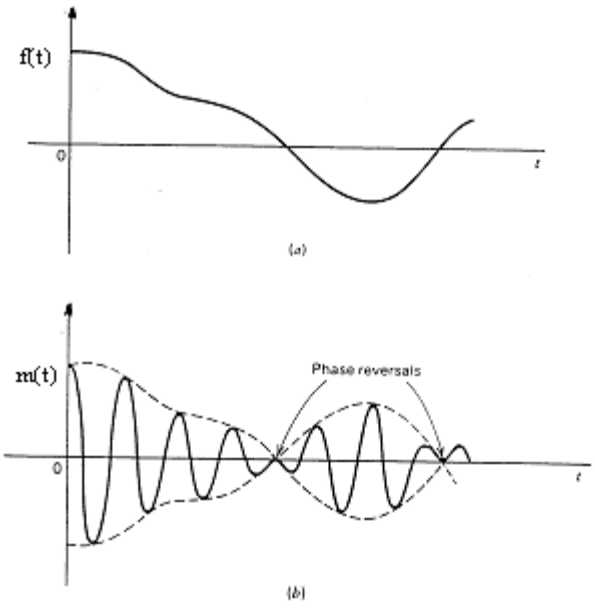
We distinguish four major classes of amplitude modulation systems: *Double Sideband Suppressed Carrier Systems* (DSB-SC), *Single Sideband Suppressed Carrier Systems* (SSB-SC), *Double Sideband Large Carrier Systems* (DSB- LC), and *Vestigial Sideband Suppressed Carrier Systems* (VSB-SC).

2.6 Double Sideband Suppressed Carrier System (DSB-SC)

Basically *double sideband suppressed carrier* (DSB-SC) modulation arises by multiplying the information signal $f(t)$ and the carrier wave $c(t)$, as follows:

$$m(t) = c(t).f(t) = A_c \cos(\omega_c t) f\ (t) \tag{4.20}$$

Consequently, the modulated signal $m(t)$ undergoes a *phase reversal* whenever the formation signal $f(t)$ crosses zero as indicated in Figure 4.4. The envelope of a DSB-SC modulated signal is therefore different from the information signal.



(a) Baseband signal. (b) DSB-SC modulated wave

Figure 4.4 The information signal f(t) and modulated signal m(t)

The Fourier transform of $m(t)$ can easily be determined from the Fourier transform properties of Table 4.2. That is,

$$M(\omega) = \frac{1}{2} A_c [F(\omega - \omega_c) + F(\omega + \omega_c)] \tag{4.21}$$

For the case when the baseband signal $f(t)$ is bandlimited to the interval $[-\omega_m, \omega_m]$, as in Figure 4.5(a), we thus -find that the spectrum of $m(t)$ is as illustrated in Figure 4.5(b). Except for a change in scale factor, the modulation process simply *translates* the spectrum of the baseband signal by $\pm\omega_c$. Note that the transmission bandwidth required by DSB-SC modulation is equal to $2\omega_m$, where (ω_m stands for the maximum frequency content of the signal $f(t)$.

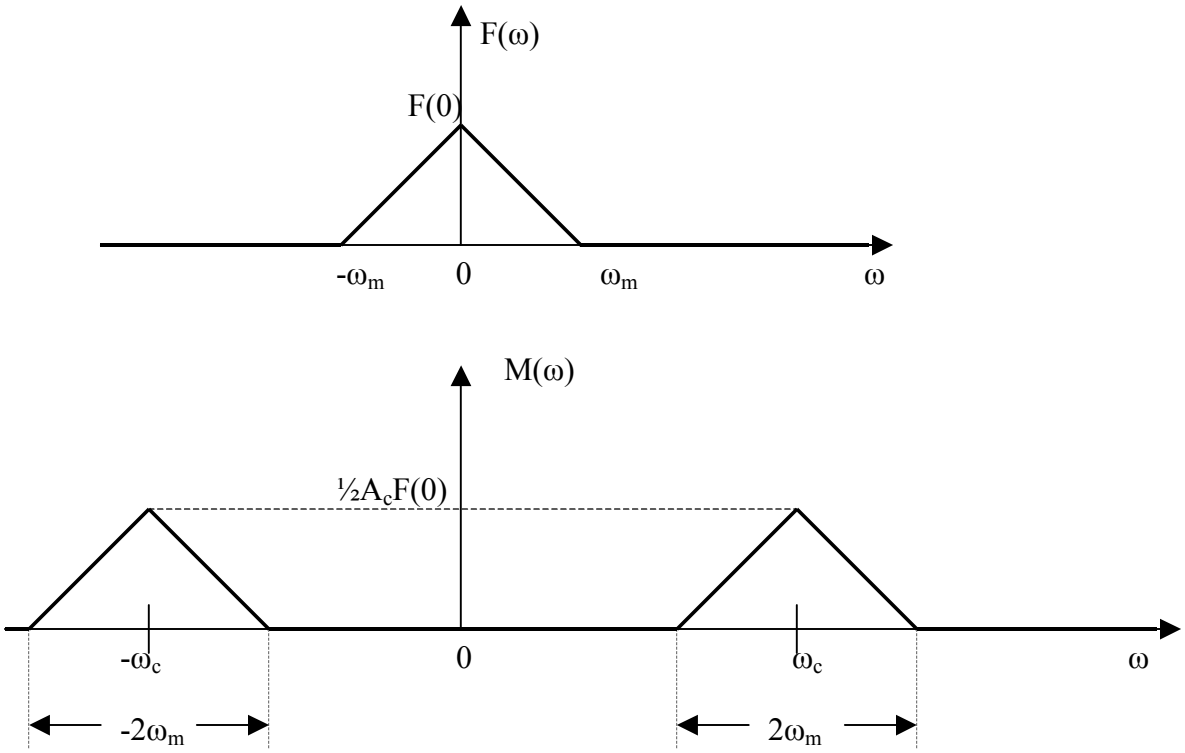


Figure 4.5 (a) Spectrum of baseband signal. (b) Spectrum of DSB-SC modulated wave

The baseband signal $f(t)$ can be uniquely recovered from a DSB-SC wave $m(t)$ by first multiplying $m(t)$ with a locally generated sinusoidal wave and then low pass filtering the product, as in Figure 4.6. It is assumed that the local oscillator signal is exactly coherent or synchronized, in both frequency and phase, with the carrier wave $c(t)$, used in the product modulator to generate $f(t)$. This method of demodulation is known as *coherent detection* or *synchronous demodulation.*

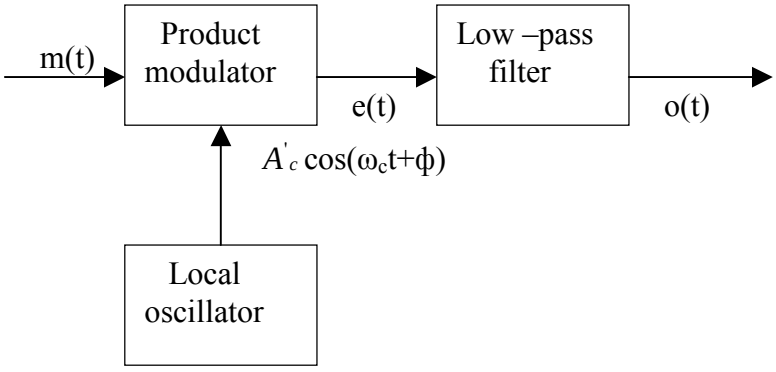


Figure 4.6 Coherent detection of DSB-SC modulated wave.

It is instructive to derive coherent detection as a special case of the more general demodulation process using a local oscillator signal of the same frequency but arbitrary phase difference ϕ, measured with respect to the carrier wave $c(t)$. Thus, denoting the local oscillator signal by $A'_c \cos(\omega_c t + \phi)$ and using equation (4.1) for the DSB-SC wave, we find that the product modulator output in Figure 4.7 is

$$\begin{aligned} e(t) &= A'_c \cos(\omega_c t + \phi) m(t) \\ &= A_c A'_c \cos(\omega_c t)\cos(\omega_c t + \phi) f(t) \\ &= \frac{1}{2} A_c A'_c \cos(2\omega_c t + \phi) f(t) + \frac{1}{2} A_c A'_c \cos\phi f(t) \end{aligned} \tag{4.22}$$

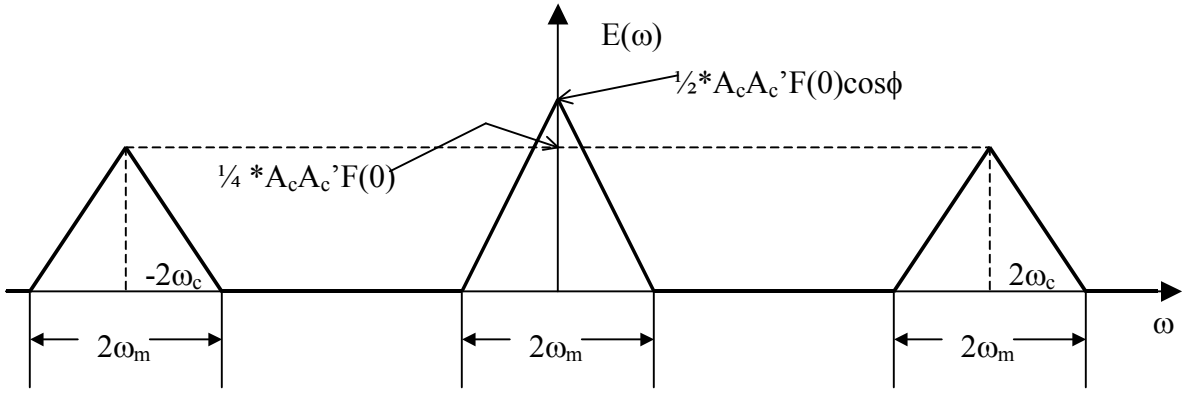


Figure 4.7 Illustrating the spectrum of a product modulator out put with a DSB-SC modulated wave as input

The first term in equation (4.22) represents the DSB-SC modulated signal with a carrier frequency of $2\omega_c$, whereas the second term is proportional to the baseband signal $f(t)$. This is further illustrated by the spectrum (Fourier Transform) $E(\omega)$ shown in Figure 4.7, where it is assumed that the baseband signal $f(t)$ is limited to the interval $[-\omega_m, \omega_m]$. It is therefore apparent that the first term in equation (4.22) is removed by the low-pass filter in Figure 4.6, provided that the cut-off frequency of this filter is greater than ω_m but less than $2\omega_c - \omega_m$. This is satisfied by choosing $\omega_c > \omega_m$. At the filter output we then obtain a signal given by

$$o(t) = \frac{1}{2} A_c A'_c \cos(\phi) f(t) \tag{4.23}$$

The demodulated signal $o(t)$ is therefore proportional to $f(t)$ when the phase error ϕ is a constant. The amplitude of this demodulated signal is maximum when $\phi = 0$, and is minimum

(zero) when $\phi = \pm\frac{\pi}{2}$. The zero demodulated signal, which occurs for $\phi = \pm\frac{\pi}{2}$, represents the *quadrature null effect* of t.he coherent detector. Thus the phase error ϕ in the local oscillator causes the detector output to be attenuated by a factor equal to $\cos\phi$. As long as the phase error ϕ is a constant, the detector output provides an undistorted version of the original baseband signal $f(t)$. In practice, however, we usually find that the phase error ϕ varies randomly with time, due to random variations in the communications channel. The result is that at the detector output, the multiplying factor $\cos\phi$ also varies randomly with time, which is obviously undesirable. Therefore, provision must be made in the system to maintain the local oscillator in the receiver in perfect synchronism, in both frequency and phase, with the carrier wave used to generate the DSB-SC modulated signal in the transmitter.

2.7 Quadrature-Carrier Multiplexing

The quadrature null effect of the coherent detector may also be put to good use in the construction of the so-called *quadrature-carrier multiplexing* or *quadrature-amplitude modulation* (QAM). This scheme enables two DSB-SC modulated waves (resulting from the application of two physically *independent* information signals) to occupy the same channel bandwidth, and yet it allows for the separation of two information signals at the receiver output. It is therefore a *bandwidth conservation scheme.*

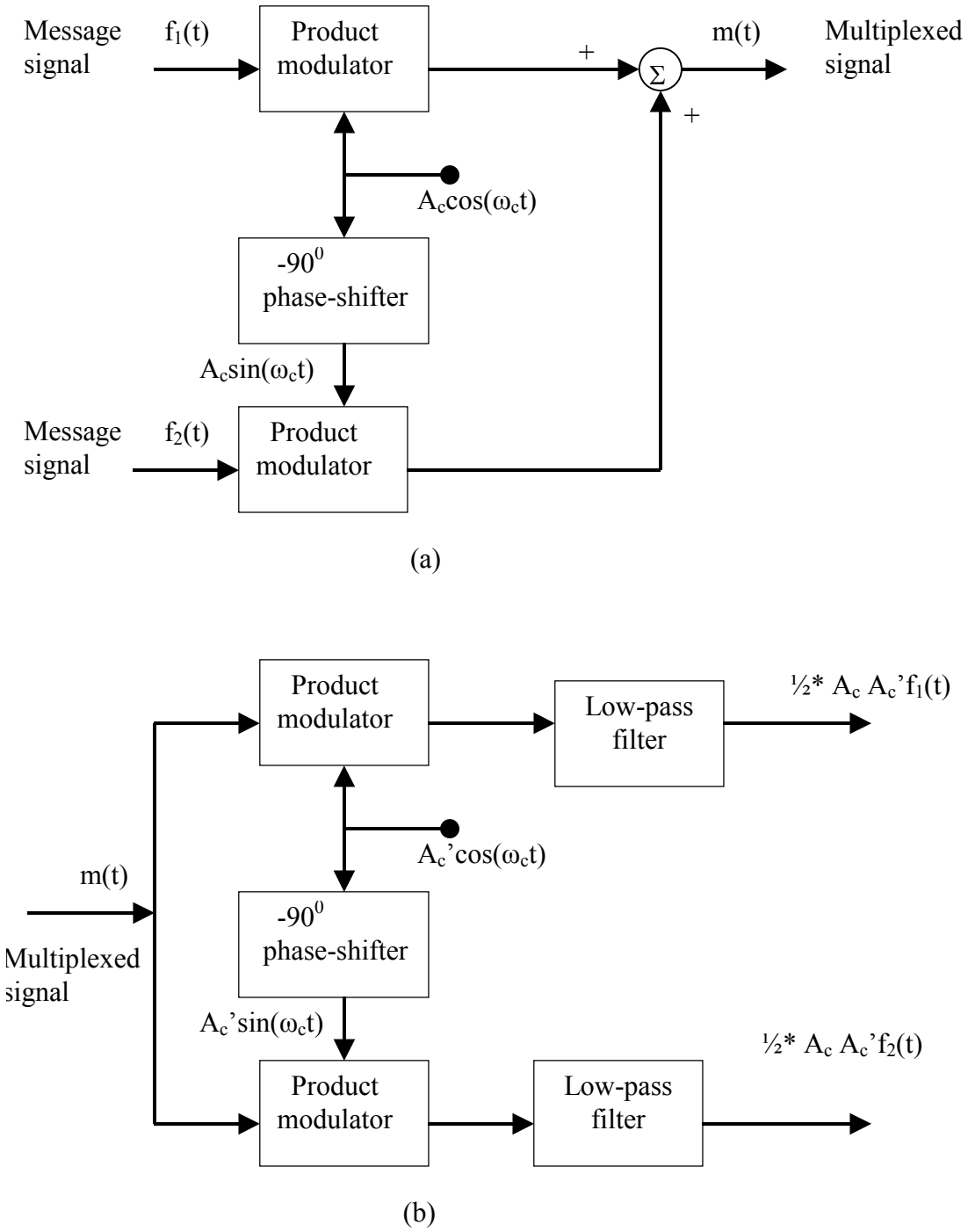


Figure 4.8 Quadrature-carrier multiplexing system.
(a) Transmitter. (b) Receiver.

A block diagram of the quadrature-carrier multiplexing system is shown in Figure 4.8. The transmitter part of the system shown in Figure 4.8(a), involves the use of two separate product modulators that are supplied with two carrier waves of the same frequency but differing in phase by -90 degrees. The transmitted signal $m(t)$ consists of the sum of these two product modulator outputs, as shown by

$$m(t) = A_c f_1(t)\cos(\omega_c t) + A_c f_2(t)\sin(\omega_c t) \tag{4.24}$$

where $f_1(t)$ and $f_2(t)$ denote the two different information signals applied to the product modulators. Thus $m(t)$ occupies a channel bandwidth of $2\omega_m$ centered at the carrier frequency ω_c, where ω_m is the maximum frequency content of the signals $f_1(t)$ and $f_2(t)$.

The receiver part of the system is shown in Figure 4.8(b) .The multiplexed signal $m(t)$ is applied simultaneously to two separate coherent detectors that are supplied with two local carriers of the same- frequency but differing in phase by -90 degrees. The output of the top detector is $\frac{1}{2}A_c A'_c f_1(t)$, whereas the output of the bottom detector is $\frac{1}{2}A_c A'_c f_2(t)$. For the system to operate satisfactorily it is important to maintain the correct phase and frequency relationships between the local oscillators used in the transmitter and receiver parts of the system.

To maintain this synchronization, we may use a Costas receiver. Another commonly used method is to send a *pilot signal* outside the passband of the modulated signal. In the latter method, the pilot signal typically consists of a low-power sinusoidal tone whose frequency and phase are related to the carrier wave $c(t)$*;* at the receiver the pilot signal is extracted by means of a suitably tuned circuit and then translated to the correct frequency for use in the coherent detector.

2.8 The Single Sideband Suppressed Carrier System (SSB-SC)

The DSB-SC system discussed so far sends the signal $m(t)$ over the channel with Fourier transform shown in Figure 4.9. One important observation regarding the Fourier transform of $m(t)$ is that is consists of two sidebands, the upper sideband and the lower sideband. There is no need to send both sidebands over the channel, since the FT of the information signal $f(t)$ can be constructed from either the upper or the lower sideband. The single sideband suppressed carrier (SSB-SC), that we intend to discuss in the following, takes advantage of the above observation by sending only one of the sidebands over the channel.

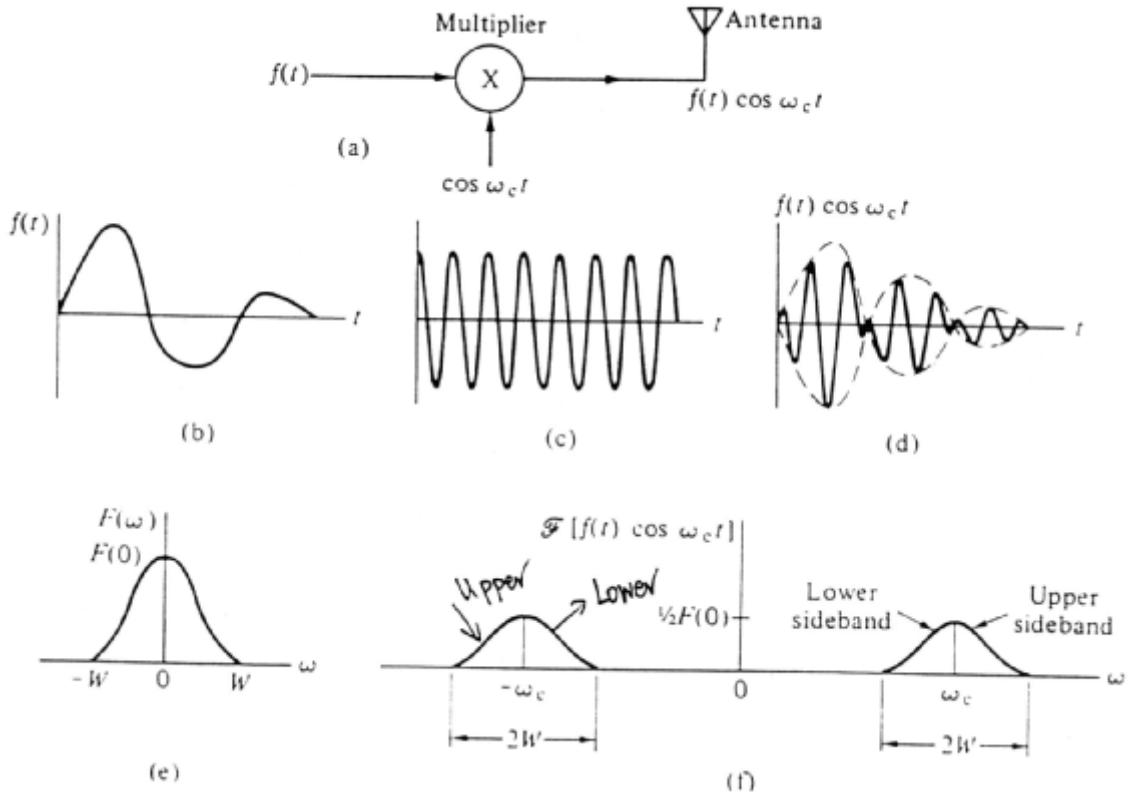


Figure 4.9 An amplitude modulation suppressed-carrier transmission

Hence, the SSB-SC system differs from the DSB-SC system because just before the modulated wave $m(t)$ is ready to head-off for the channel, one of the sidebands is chopped off. The block diagram of an SSB-SC system is shown in Figure 4.10. Furthermore, the sequence of Fourier transform plots shown in Figure 4.11 (upper sideband) demonstrate that the system works. The only problem with the block diagram of the system shown in Figure 4.10 is that it requires the utilization of a "perfect filter" that eliminates completely the frequencies a little bit below ω_c and passes without distortion the frequencies a little bit above ω_c (we are referring only to the positive frequencies of the upper sideband signal). To relax the requirement for such a "perfect" filter we will consider another way of generating the upper sideband version of the signal $m(t)$.

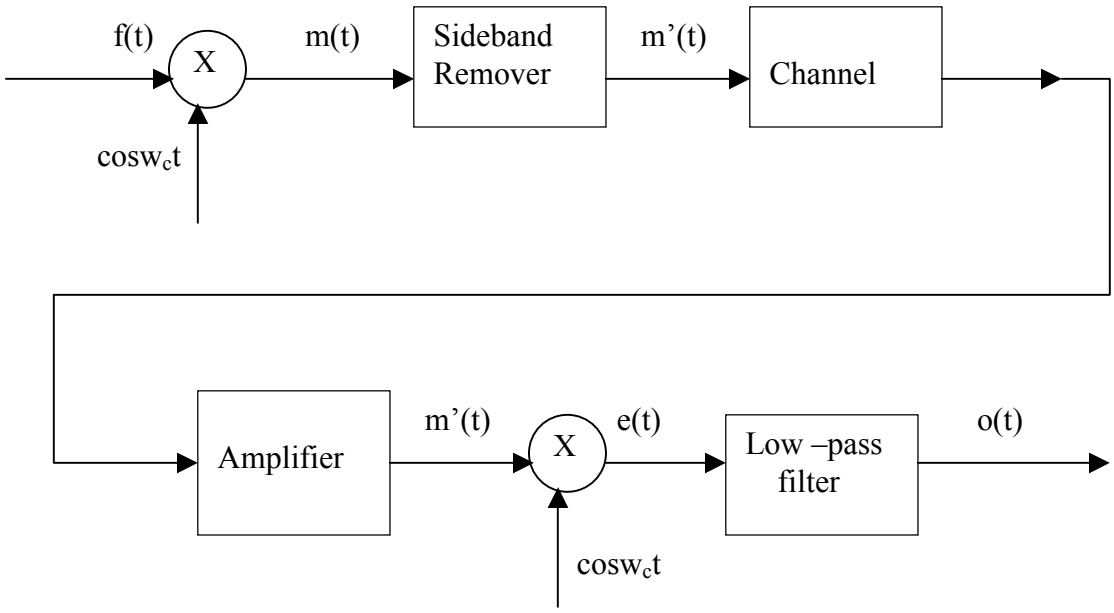


Figure 4.10 Block diagram of a SSB-SC system

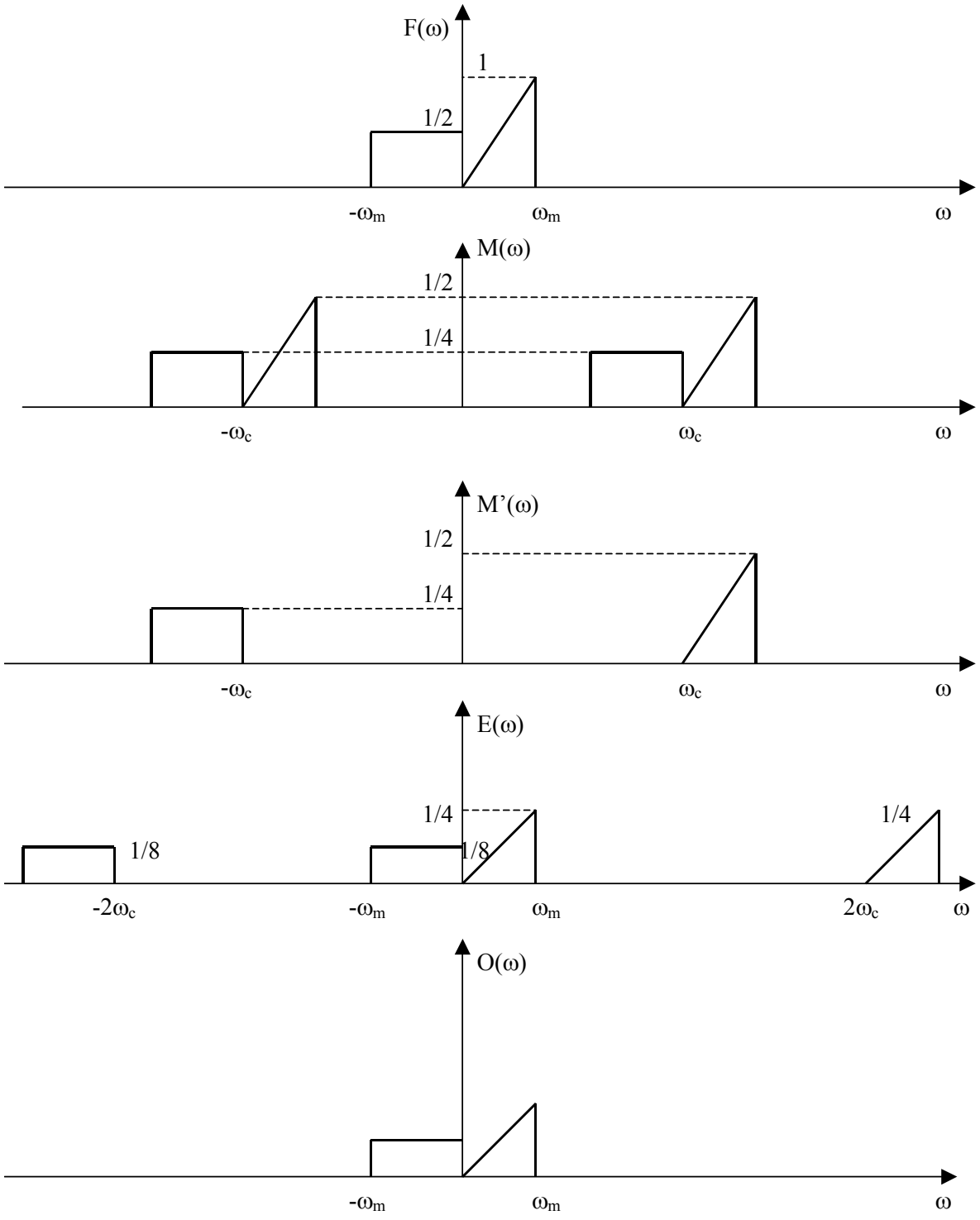


Figure 4.11: FT's at various stage of the SSB-SC system of Figure 4.10

For the sake of simplicity consider first the case where the information signal is sinusoidal (i.e., $f(t)=\cos(\omega_m t)$. Then, the Fourier transform of $m(t)=\cos(\omega_m t)\cos(\omega_c t)$ is given in Figure 4.12. From Figure 4.12, it is easy to identify the upper sideband of $m(t)$ in the frequency domain; also, the time-domain representation of this upper sideband is readily defined as follows:

$$m_+(t)=\frac{1}{2}\cos(\omega_c+\omega_m)t \tag{4.25}$$

or

$$m_+(t)=\frac{1}{2}\cos(\omega_c t)\cos(\omega_m t)-\frac{1}{2}\sin(\omega_c t)\sin(\omega_m t) \tag{4.26}$$

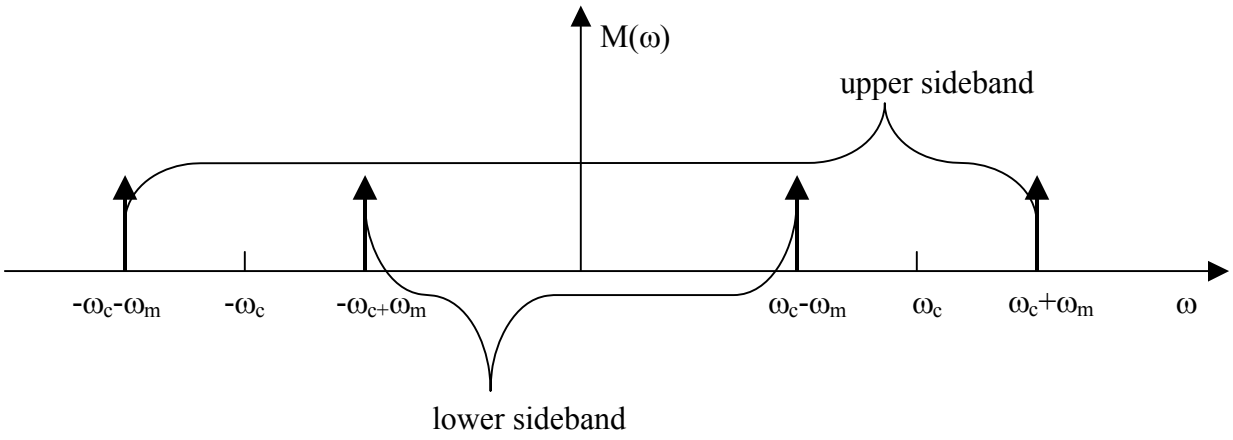


Figure 4.12 Fourier transform of m(t) = cos(ωm+ωc)t

In Figure 4.13 we depict a way of constructing the upper sideband signal of equation (4.26) without having to resort to the design of perfect filters. In Figure 4.13 there is a system designated by a block with the -90 notation inside it. We know what this block does to an input signal that is sinusoidal; it shifts the phase of the sinusoidal signal by -90 degrees. We want to be able to characterize this block in the frequency domain. To do so let us identify the Fourier transforms of an input signal to this block, equal to $\cos(\omega_m t)$ and the output signal produced by this block, equal to $\cos(\omega_m t-90)=\sin(\omega_m t)$. Obviously,

$$FT[\sin(\omega_m t)]=-j\pi\delta(\omega-\omega_m)+j\pi\delta(\omega+\omega_m) \tag{4.27}$$

$$FT[\cos(\omega_m t)]=\pi\delta(\omega-\omega_m)+\pi\delta(\omega+\omega_m) \tag{4.28}$$

Based on the above equations it is straightforward to derive the result that

$$FT[\sin(\omega_m t)] = -jsignum(\omega)FT[\cos(\omega_m t)] \tag{4.29}$$

where the function $signum(\omega)$ is equal to 1 when $\omega > 0$ and equal to -1 when $\omega < 0$. Equation (4.29) tells us that the transfer function of the block designated by the notation -90 is equal to $-jsignum(\omega)$. Now we are ready to generalize the construction of an upper sideband signal using the block diagram of Figure 4.13 for the case where the information signal $f(t)$ is of arbitrary nature (i.e., not necessarily sinusoidal). In Figure 4.14, we provide a block diagram for the construction of the upper sideband signal of a modulated wave $m(t)$ when the information signal $f(t)$ is arbitrary. Comparing Figures 4.13 and 4.14 we see that they are identical, where in both figures the block designated by the notation -90 corresponds to a system with transfer function equal to $-jsignum(\omega)$.

In Figure 4.15 we show, in a pictorial fashion, why the block diagram of Figure 4.14 works. What is worth noting is that the output of a -90 degree shift system is called the *Hilbert transform* of the input. Furthermore, if we designate by $f(t)$ the input to a -90 degree shift system the output (Hilbert transform) of this system is designated by $\hat{f}(t)$. Finally, it is easy to show that

$$f(t) = f(t) * \frac{1}{\pi t} \tag{4.30}$$

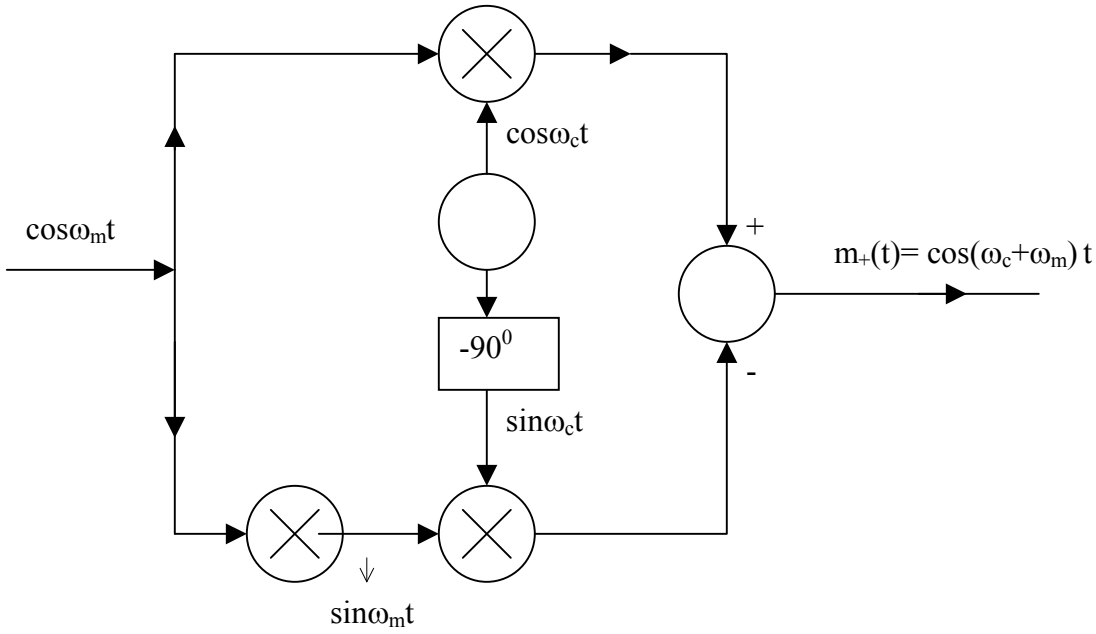


Figure 4.13 Block diagram of a system that generates the upper sideband of m(t) ($f(t)=\cos\omega_m t$, $c(t) = \cos\omega_c t$)

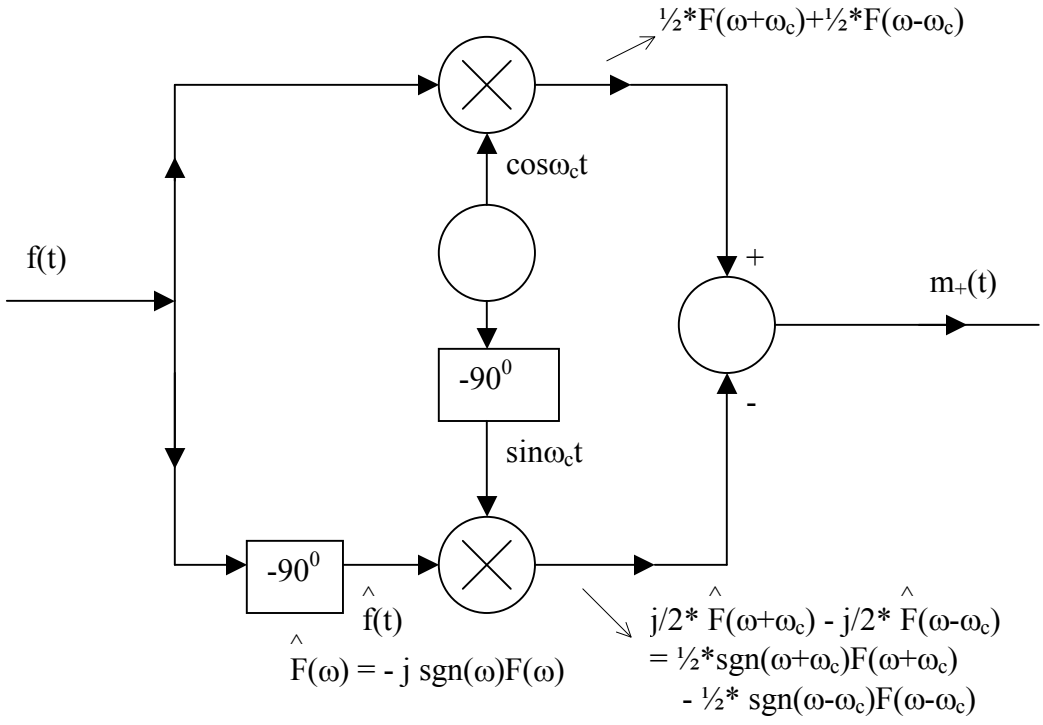


Figure 4.14 Block diagram of a system that generates the upper sideband of m(t)

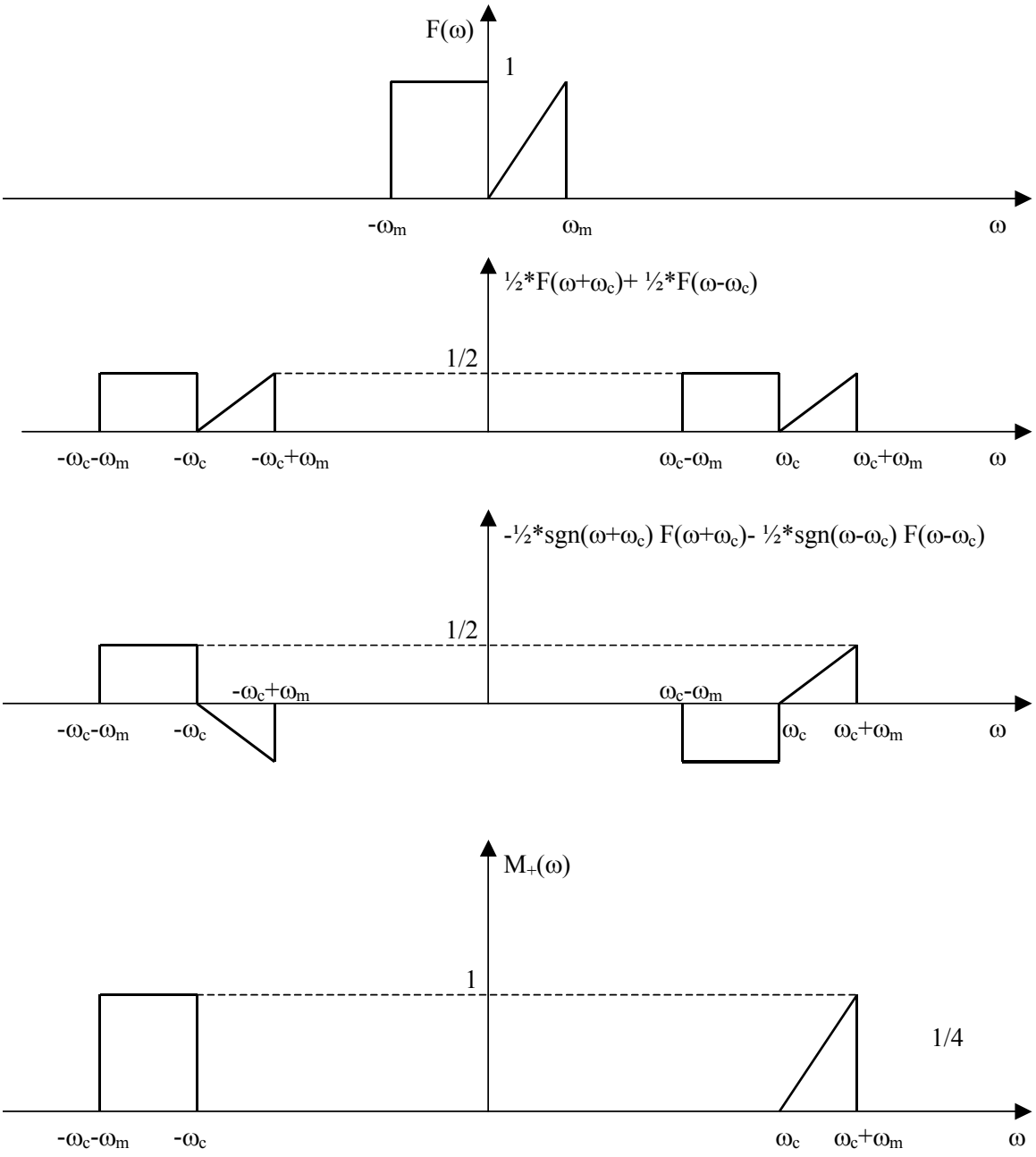


Figure 4.15: FT at various stages of the block diagram of Figure 4.14

2.9 Double Sideband Large Carrier (DSB-LC) System

The main idea behind this system is money. Its claim to fame comes from the use of a very cheap receiver called peak or envelope detector. To understand this detector let us first look at the modulated signal $m(t)$ for two examples of information signals $f(t)$. In Figure 4.16 we show $m_1(t)$ for an information signal $f_1(t)$ and $m_2(t)$ for an information signal $f_2(t)$. The difference between the two information signals $f_1(t)$ and $f_2(t)$ is that $f_1(t)$ is always positive, while $f_2(t)$ assumes positive and negative values. If we now look at the positive peaks of $m_1(t)$ we see that a line through them produces the information signal $f_1(t)$. On the other hand if we look at the positive peaks of $m_2(t)$ a line through them produces the information signal $f_2(t)$ only when $f_2(t)$ is positive and it produces the negative of $f_2(t)$ when the signal is negative. The reason that the positive peaks of $m_1(t)$ and $m_2(t)$ are important is because we can design an inexpensive receiver that traces these positive peaks. Hence, when the signal $f(t)$ is always positive, this receiver will be able to reproduce the information signal $f(t)$ from $m(t)$*;* this is not the case though when the signal $m(t)$ assumes positive and negative values. Unfortunately, information signals can be positive or negative. To remedy the problem of a negative information signal we can add a constant to the signal that is larger than or equal to the most negative signal value. Hence, now

$$m(t) = [A + f(t)]\cos(\omega_c t) = A\cos(\omega_c t) + f(t)\cos(\omega_c t) \qquad (4.31)$$

The Fourier transform of $m(t)$ is shown in Figure 4.17 and it is provided by the following expression.

$$M(\omega) = A\pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] + \frac{1}{2}[F(\omega - \omega_c) + F(\omega + \omega_c)] \qquad (4.32)$$

By looking at Figure 4.17 we can justify the name DSB-LC for this communication system. The Fourier transform of the modulated wave $m(t)$ has both of the sidebands (upper and lower). Furthermore, the presence of the pure carrier $A\cos(\omega_c t)$ is evident by the presence of the two impulses in the Fourier transform of $m(t)$.

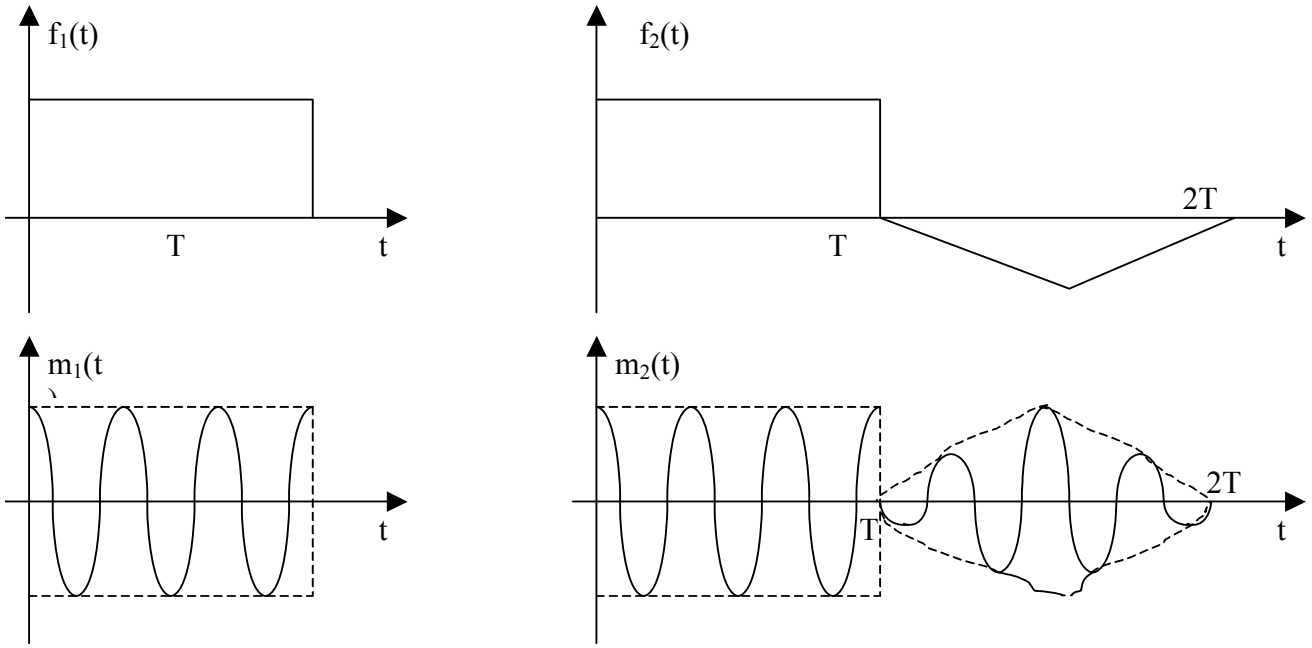


Figure 4.16 DSB-LC Signals for two information signals $f_1(t)$ and $f_2(t)$

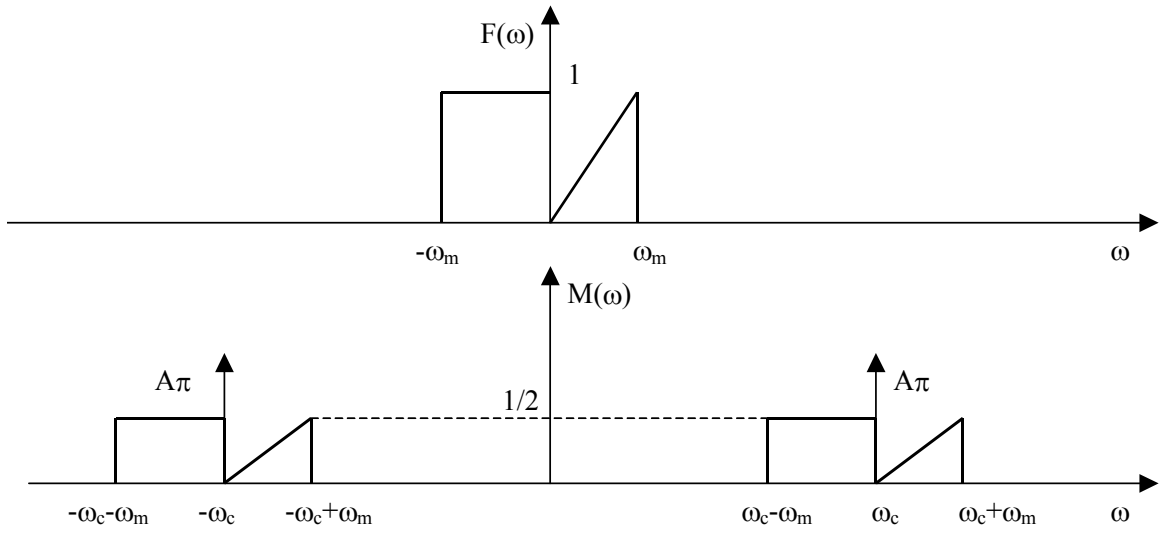


Figure 4.17: FT of a DSB-LC signal

Consider now the special case where the information signal $f(t)$ is of the form.

$$f(t) = K\cos(\omega_m t) \tag{4.33}$$

Based on our previous discussion, the maximum amplitude of the pure carrier that we need to add to the DSB-SC modulated signal to make it a legitimate DSB-LC signal must satisfy the following inequality:

$$A \geq K \tag{4.34}$$

Then,

$$m(t) = A\left(1 + \frac{K}{A}\cos(\omega_m t)\right)\cos(\omega_c t) \tag{4.35}$$

We usually define

$$m = \frac{K}{A} \tag{4.36}$$

the modulation index of the system. Since m's maximum value is one and its minimum value is zero, it is often given in percent. In Figure 4.18 we show the DSB-LC signal for the case of an information signal that is sinusoidal and for various m values (i.e., $m = 1$, $m < 1$ and $m > 1$). The case $m > 1$ is not allowed because then we are not going to be able to recover the information signal $f(t)$ .In Figure 4.19 we show a modulated wave for an arbitrary information signal $f(t)$ and a modulation index $m < 1$ or $m > 1$.

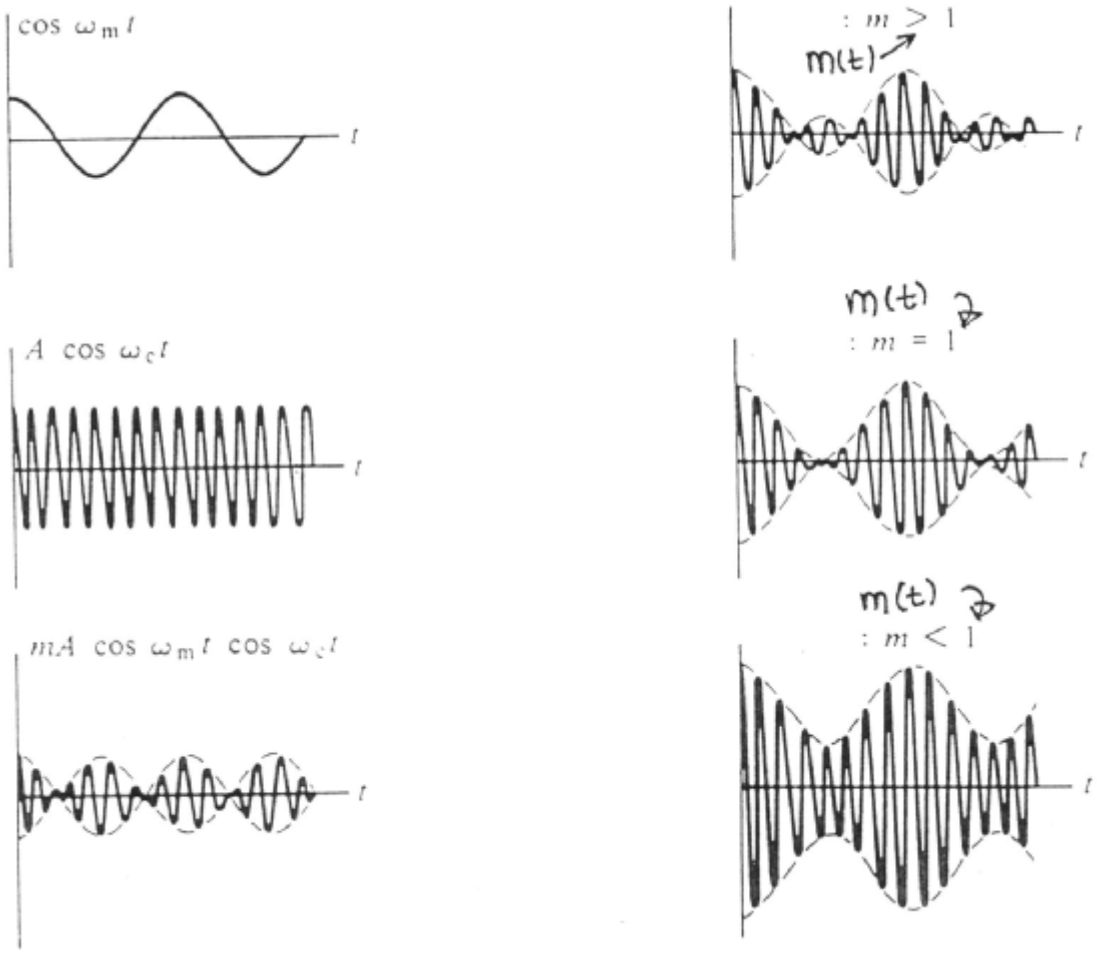


Figure 4.18 Effects of varying modulation indixes

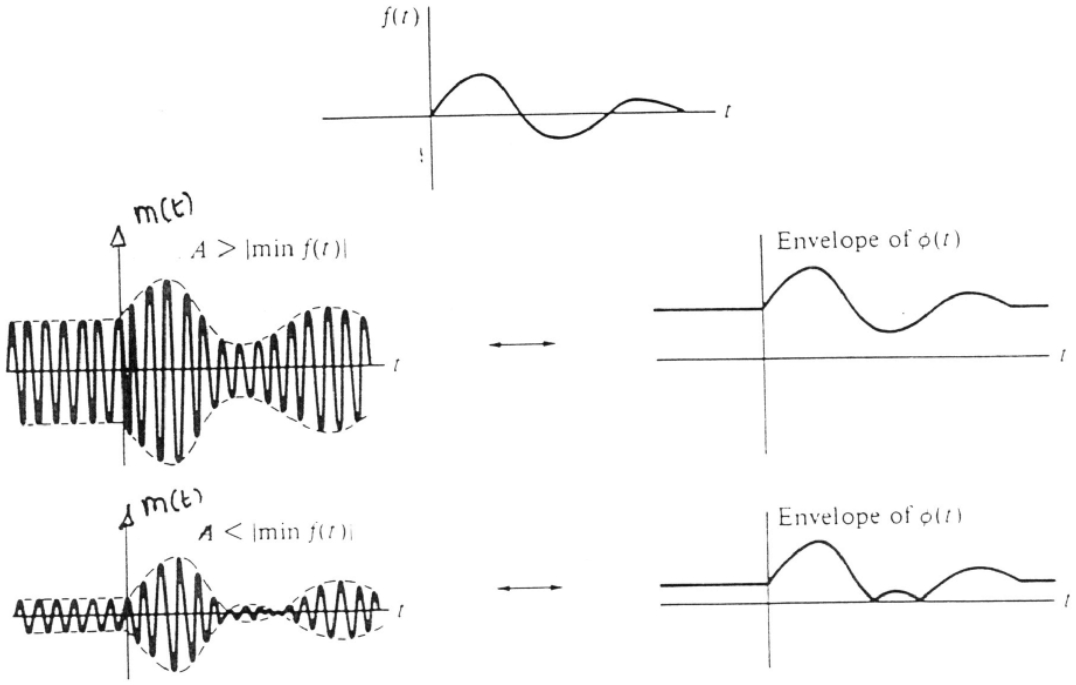


Figure 4.19 Importance of sufficient carrier in DSB-LC waveform

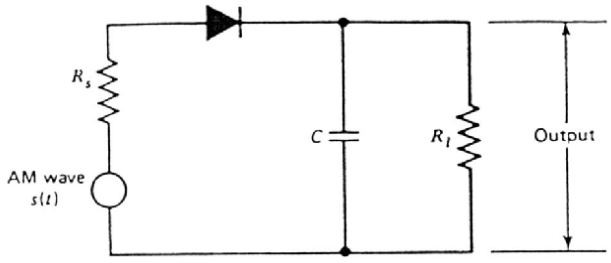
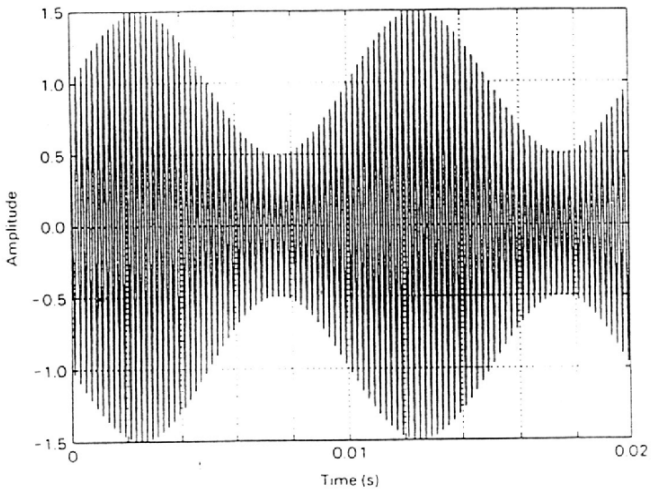
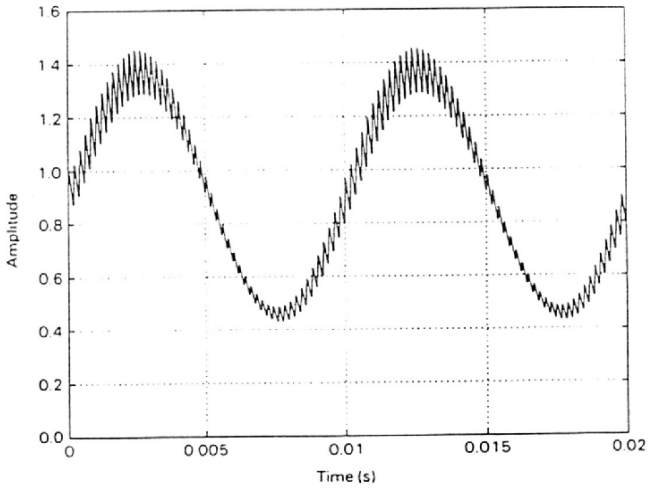


Figure 4-20 Envelop Detector
(a) Circuit Diagram



(b)



(c)

Figure 4.20 Envelop detector.
(a) Circuit diagram. (b) AM wave input (c) Envelop detector output

As we emphasized before, the information signal $f(t)$ can be recovered from a DSB-LC signal by utilizing a peak (envelope) detector. An envelope detector consists of a diode and a resistor-capacitor (RC) filter (see Figure 4.20). The operation of this envelope detector is as follows. On a positive half-cycle of the input signal, the diode is forward biased and the capacitor C charges up rapidly to the peak value of the input signal. When the input signal falls below this value, the diode becomes reverse-biased and the capacitor C discharges slowly through the load resistor R_l. The discharging process continues until the next positive half-cycle. When the input signal becomes greater than the voltage across the capacitor, the diode conducts again and

the process is repeated. We assume that the diode is ideal, presenting resistance r_f to current flow in the forward-biased region, and infinite resistance in the reverse-biased region. We further assume that the AM wave applied to the envelope detector is supplied by a voltage source of internal impedance R_s. The charging time constant $(r_f + R_s)C$ must be short compared with the carrier period $2\pi/\omega_c$, that is

$$(r_f + R_s)C << \frac{2\pi}{\omega_c} \tag{4.37}$$

so that the capacitor C charges rapidly and thereby follows the applied voltage up to the positive peak when the diode is conducting. On the other hand, the discharging time constant $R_l C$ must be long enough to ensure that the capacitor discharges slowly through the load resistor R_l, between positive peaks of the modulated wave, but not so long that the capacitor voltage will not discharge at the maximum rate of change of the modulating wave, that is

$$\frac{2\pi}{\omega_c} << R_l C << \frac{1}{\omega_m} \tag{4.38}$$

where ω_m is the information signal bandwidth. The result is that the capacitor voltage or detector output is nearly the same as the envelope of the AM wave, as shown in Figure (4.20).

One of the serious flaws of a DSB-LC system is that it wastes transmitter power. To illustrate that consider the DSB-LC signal of equation (4.31). It is easy to show that

$$\begin{aligned}\text{Power of } m(t) = {} & \text{Power of } \{A\cos(\omega_c t)\} + \text{Power of } \{f(t)\cos(\omega_c t)\} + \\ & \text{Power of } \left\{\sqrt{2Af(t)\cos(\omega_c t)}\right\}\end{aligned} \tag{4.39}$$

The power of $A\cos(\omega_c t)$ is equal to $A^2/2$, while the power of $f(t)\cos(\omega_c t)$ can be shown to be equal to the one-half the power of $f(t)$; the latter result is valid under the legitimate assumption that the bandwidth of the information signal is much smaller than the carrier frequency ω_c. Finally, the power of $\sqrt{2Af(t)\cos(\omega_c t)}$ is equal to zero under the assumptions that (i). the DC value of *f(t)* is zero and (ii) the bandwidth of the information signal is much smaller than the carrier frequency. Consequently,

$$\text{Power of } m(t) = \frac{A^2}{2} + \frac{\text{Power of } f(t)}{2} \tag{4.40}$$

For the special case where $f(t) = K\cos(\omega_m t)$ we can conclude that

$$\text{Power of } m(t) = \frac{A^2}{2} + \frac{A^2 m^2}{4} = \frac{A^2}{2}\left(1 + \frac{m^2}{2}\right) \tag{4.41}$$

It is worth noting that from the power dedicated to transmit the modulated signal, $A^2/2$ goes to the transmission of the pure carrier and the rest goes to the transmission of the useful signal. If we define the *efficiency* of our system as the ratio of the power dedicated to the transmission of the useful signal versus the power dedicated to the transmission of the modulated signal we conclude that the efficiency of our DSB-LC system (for a sinusoidal information signal) is equal to

$$\text{Efficiency} = \eta = \frac{m^2}{m^2 + 2} \tag{4.42}$$

The best we can do is to let η assume its maximum value; this happens when $m = 1$. Then,

$$\eta = 0.33 \ \ (33\%) \tag{4.43}$$

So 67% of the available transmitter power goes to the carrier (alarming quantity!) Usually, we do not know K so we might operate at less than 33 % efficiency.

The notion of the modulation index can be extended to all signals, as long as they are normalized so that their maximum negative value is unity. To be more specific consider the DSB-LC signal corresponding to an information signal whose maximum negative value is different than unity. Then,

$$\begin{aligned} m(t) &= A\cos(\omega_c t) + f(t)\cos(\omega_c t) \\ &= A\cos(\omega_c t) + |\min\{f(t)\}| \frac{f(t)}{|\min\{f(t)\}|}\cos(\omega_c t) \\ &= A\left[1 + \frac{|\min\{f(t)\}|}{A} f'(t)\right]\cos(\omega_c t) \end{aligned} \tag{4.44}$$

where $f'(t)$ is the normalized version of $f(t)$. Note that $f'(t)$ has minimum value of -1. In the above equation we define the modulation index m to be equal to the ratio

$$\frac{|\min\{f(t)\}|}{A} \tag{4.45}$$

Actually, the aforementioned definition is applicable independently of whether $f(t)$ has minimum value of unity or not.

2.10 Vestigial Sideband Suppressed Carrier (VSB-SC)

The stringent frequency-response requirements on the sideband remover of a SSB-SC system can be relaxed by allowing a part, called *vestige,* of the unwanted sideband to appear at the

output of the modulator. Thus, we simplify the design of the sideband filter at the cost of a modest increase in the channel bandwidth required to transmit the signal. The resulting signal is called *vestigial/-sideband suppressed carrier* (VSB-SC) system. The suppressed carrier name sterns from the fact that one more time no pure carrier is sent over the channel.

To generate a VSB-SC signal, we begin by generating a DSB-SC signal and passing it through a sideband filter with frequency response $H(\omega)$ as shown in Figure 4.21. In the time domain the VSB-SC signal may be expressed as

$$m(t) = [f(t)\cos(\omega_c t)] * h(t) \tag{4.46}$$

where $h(t)$ is the impulse response of the VSB-SC filter. In the frequency domain the corresponding expression is

$$M(\omega) = \frac{1}{2}[F(\omega - \omega_c) + F(\omega + \omega_c)]H(\omega) \tag{4.47}$$

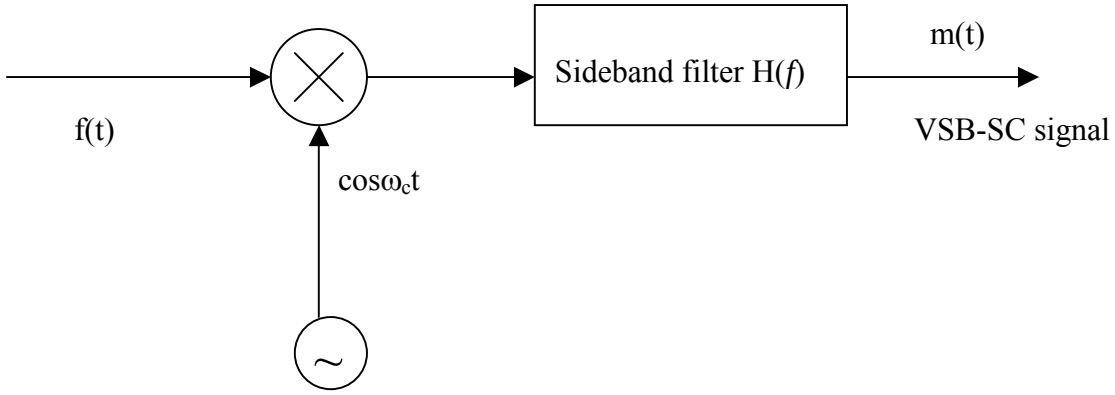


Figure 4.21 Generation of a VSB-SC signal

To determine the frequency-response characteristics of the filter, let us consider the demodulation of the VSB-SC signal $m(t)$. We multiply $m(t)$ by the carrier component $\cos(\omega_c t)$ and pass the result through an ideal lowpass filter, as shown in Figure 4.22. Thus the product signal is

$$e(t) = m(t)\cos(\omega_c t) \tag{4.48}$$

or, equivalently

$$E(\omega) = \frac{1}{2}[M(\omega - \omega_c) + M(\omega + \omega_c)] \tag{4.49}$$

If we substitute for $M(\omega)$ from equation (4.47) into equation (4.48), we obtain

$$M(\omega)=\frac{1}{4}[F(\omega-2\omega_c)+F(\omega)]H(\omega-\omega_c)$$
$$+\frac{1}{4}[F(\omega)+F(\omega+2\omega_c)]H(\omega+\omega_c) \tag{4.50}$$

The lowpass filter rejects the double frequency components and passes only the components in the frequency range $|\omega|<\omega_m$. Hence the signal spectrum at the output of the ideal lowpass filter is

$$O(\omega)=\frac{1}{4}F(\omega)[H(\omega-\omega_c)+H(\omega+\omega_c)] \tag{4.51}$$

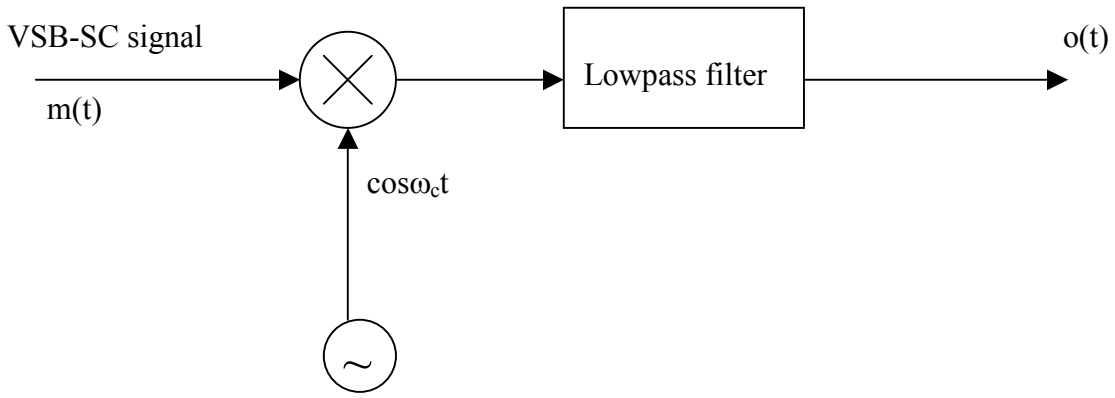


Figure 4.22 Demodulation of the VSB-SC signal m(t)

We require that the information signal at the output of the lowpass filter be undistorted. Therefore the VSB-SC filter characteristic must satisfy the condition:

$$H(\omega-\omega_c)+H(\omega+\omega_c)=\text{constant} \quad |\omega|\le\omega_m \tag{4.52}$$

The condition is satisfied by a filter that has the frequency response characteristic shown in Figure 4.23. We note that $H(\omega)$ selects the upper sideband and a vestige of the lower sideband. It has an odd symmetry about the carrier frequency ω_c, in the frequency range $\omega_c-\omega_a<\omega<\omega_c+\omega_a$, where ω_a is a conveniently selected frequency that is some small fraction of ω_m, i.e., ($\omega_a<<\omega_m$. Thus, we obtain an undistorted version of the transmitted signal. Figure 4.24 illustrates the frequency response of a VSB filter that selects the lower sideband and a vestige of the upper sideband.

In practice, the VSB filter is designed to have some specified phase characteristics. To avoid distortion of the information signal, the VSB filter should be designed to have a linear phase over its passband $\omega_c - \omega_a < \omega < \omega_c + \omega_a$

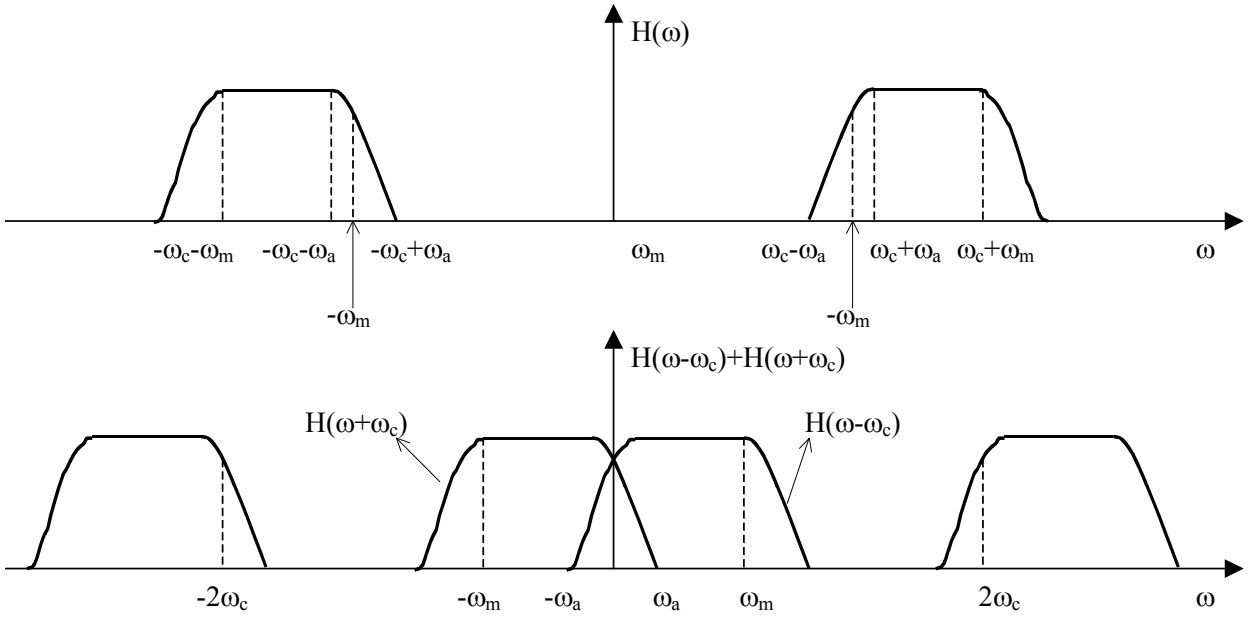


Figure 4.23: VSB filter characteristics

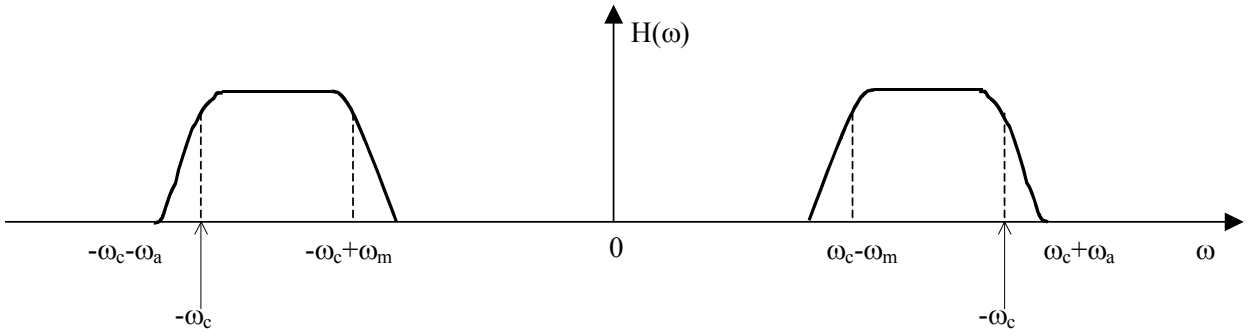


Figure 4.24: Frequency response of VSB filter for selecting the lower sideband of the message signals.

2.11 Why modulation?

Modulation is the process by which a property of a parameter of a signal in proportion to a second signal. The primary reason for using modulation in communication is:

1. To raise up the frequency of a signal to reduce the wavelength such that a relatively small antenna can transmit or receive the signal.

2. To separate base band signals in frequency (or time) so that more than one signal can be transmitted on the same channel.

3. To have the base band signal transformed for ease of transmission.

The degree to which a signal is modulated is measured by a modulation index, which will have a different physical significance for each type of modulation. Every form of modulation has advantages and disadvantages when compared to the others. FM has more strengths than weaknesses compared to the others. Delta Modulation has more weakness than strengths, but all have their applications. Some modulation techniques and their characteristics are listed in the table 4.3.

2.12 Aliasing

Aliasing occurs in most forms of modulation when the sampling rate or the carrier frequency is not large enough compared to the maximum message frequency.

Once the spectra have overlapped there is no way to separate the two signals. The nyquist theorem states that for a sampled signal, such as PAM, the sampling frequency f_s must be at least twice the maximum frequency of the signal to prevent aliasing from occurring.

3 Simulation

In this simulation we are going to see the amplitude modulation and demodulation with different parameters.

Let us consider a sinusoidal message signal of 100 Hz and a sinusoidal carrier in the transmitter of 1000 Hz.

f = 100;
f_c = 1000;
Modulation index, m = 0.5;
Sampling frequency, fs=16384.

MODULATION

1) Construct the time array as following

   t = [0:1/fs:0.05];

2) Construct the message signal as

   signal = sin(2πft)

3) Calculate the required dc offset from modulation index definition

   A = abs(min(signal))/m

4) Construct the carrier at the transmitter end as

   carrier1 = sin(2πf$_c$t);

5) Modulated output of the transmitter is given by

   x = (signal+A).*carrier1;

   (Notice a dot before the multiplication sign. In Matlab it represents multiplication element by element.)

6) Find the Fourier transform of 'signal', 'carrier1' and 'x'. Name them say 'S1', 'C1' and 'X' respectively. Use the 'ft' function for this. You have to create your own 'ft' function. The code for this function is given below. Make sure that you saved this function as a separate Matlab file with the same name 'ft.m' in the Matlab working directory.

```
function [X]=ft(x,fs)

n2=length(x);
n=2^(max(nextpow2(fs),nextpow2(n2)));
X1=fft(x,n)/fs;
X=X1(1:n/8);
```

   An example has been given below how you find the fourier transform using this Matlab function. You can write the following line of code in your main program to find fourier transform of a signal.

   [S1]=ft(signal,fs);

For good demodulation of the received signal, the carrier at the receiver end needs to be of the same frequency and phase as the carrier of the transmitter end. In practice this may not be the case.

7) Carrier at the receiver (demodulator) is given by the following equation.

carrier2=sin(2π(f_c+delta)t+phi);

Where delta is the frequency error and phi is the phase error. Initially assume that both of them are zero.

8) For demodulation multiply the received signal by the carrier and name it 'x1' and then do a low pass filtering.

For low pass filtering use the following piece of Matlab code.

b=fir1(24,500/fs); (*The first parameter gives you the order of the filter. The second parameter is the cutoff frequency. Here it is 200 Hz*.)

x_demod=conv2(x1,b,'same'); *Here 'b' is the coefficient of the filter. So to find the filtered output we did the convolution of 'x1' and 'b'.*

9) Find the Fourier transform of 'carrier2' and 'x_demod'. Name them C2 and S2.

10) Plot the following:

'**signal':** message signal before modulation versus time
'x': modulated signal versus time
'x_demod': demodulated signal versus time
'**S1':** message signal in frequency domain
'C1': carrier at transmitter end in frequency domain
'X': modulated signal in frequency domain
C2: carrier at the receiver end in frequency domain
S2: demodulated signal in the frequency domain

You may use following Matlab functions to plot all of them together.

Subplot
Plot

11) Repeat the simulation using square wave carrier at both ends. What happens if you use different wave shapes? Explain. Actually in the hardware experiments that follow you will be using a square wave carrier.

12) Repeat the above simulation with a 'sinc' function as the message signal.

13) Repeat the above simulation with phi = π/4, π/2 in step 7). What happens? Explain. What happens if delta = 200/400 Hz?

4 Pre-lab questions

Non sinusoidal carrier

Consider a case of double-sideband suppressed carrier amplitude modulation (DSB-SC AM), where instead of having a sinusoid to modulate our signal we use some other periodic signal p(t) with Fourier series expansion:

$$p(t) = \sum_{-\infty}^{\infty} P_n e^{jn\omega_0 t} \tag{4.53}$$

1. By providing a qualitative example (i.e. assume a specific p(t), say square wave that we will be using in our lab) show how the spectrum of the modulated signal will look like.

2. How can we demodulate a signal of this kind?

Explain and illustrate all your answers.

5 Implementation

By this time you already know that in amplitude modulation, the amplitude of the carrier frequency varies with the amplitude of the desired message signal. Typically the carrier will be a cosine or some sine wave but in this lab a square wave is used as the carrier for simplicity in implementation. Amplitude Modulation is implemented by multiplying the message signal with the carrier as in the theory part, which can be expressed as:

$$m(t) = f(t).c(t) = f(t).\cos(\omega_c t) \quad \text{(For a cosine carrier)} \tag{4.54}$$

The end result with sinusoidal modulation is such that the spectrum of $X(f)$ remains unchanged but is removed in frequency and centered about ω_c. For any general carrier, the multiplication of two signals in time yields convolution in frequency.

A DSB AM signal is described by the equation:

$$m(t) = f(t).c(t) = (f(t) + A)\cos(\omega_c t) \tag{4.55}$$

A is the average value of the signal, or the dc component. Typically a message signal will have no dc component.

When $A > 0$, the $m(t)$ is a DSB-LC (Large Carrier) AM signal. In this case the dc term must be added to the message signal. The case when $A = 0$ is called Suppressed Carrier (DSB-SC)

AM. The modulated index for AM is a measure of how large A is with respect to the amplitude of $f(t)$.

$$m = \frac{|f(t)_{\min}|}{A} \tag{4.56}$$

Where $f(t)_{\min}$ is the minimum value of the message signal before any dc is added and A is the amount of dc added. Note that $m = 100\%$ for $A = |f(t)_{\min}|$.

There are two forms of **demodulation** used in this lab:

1. Synchronous: The modulated waveform is multiplied by another carrier of (hopefully) identical frequency and phase to the AM carrier. High frequency terms that result from the multiplication are removed with a filter and only the message is left. Note that the carrier must be regenerated for an actual receiver.

2. Envelop Detection: When $m < 100\%$, peak detection (sample and hold) can be used. Sample and hold means that the amplitude of the signal at a given point in time is sampled and held until the next sample pulse arrives.

The FET used in the AM modulator and demodulator (Figure 4.25 and Figure 4.26) is used as a variable resistor, where the resistance between the drain (D) and the source (S) varies proportional to the voltage applied to the gate (G).

Since there is a square wave varying from 0 to $-V_{cc}$ volts been applied to the gate, the resistance across the source and the drain can be thought as either an open ($R = \infty$) when $V_g = -V_{cc}$ or a short ($R = 0$) for $V_g = 0$. Analysis of the circuit shows a gain of $V_{out}/V_{in} = 1$ for $R = \infty$ (differential amplifier configuration) and a gain of $V_{out}/V_{in} = -1$ when $R = 0$ (inverting amplifier configuration).

6 Procedure

6.1 AM Modulation

1) Build the modulator shown in Figure 4.25 and Figure 4.26.

2) Connect the oscilloscope to the function generator output (or the message oscillator output), $V_{in}(t)$ and to the modulator output, $V_o(t)$, using direct coupling (DC) throughout. Set $V_{in}(t)$ to a sinewave with $f_m = 100$ Hz.

3) Adjust the dc offset and amplitude of the message of the message oscillator to yield a DSB-LC modulated waveform having a modulation index $m = .75$. Sketch $V_{in}(t)$ and $V_o(t)$. Change $V_{in}(t)$ to a triangular waveform and sketch $V_{in}(t)$ and $V_o(t)$.

4) Adjust the dc offset of the message oscillator to zero, thus producing DSB-SC. Again sketch $V_{in}(t)$ and $V_o(t)$ first for $V_{in}(t)$ sinusoid and then $V_{in}(t)$ triangular.

5) Obtain once again the DSB-SC signal for $V_{in}(t)$ sinusoidal and $f_m = 100$ Hz. Observe and sketch the spectra of $V_{in}(t)$ and $V_o(t)$, using the spectrum analyzer. To see the modulated output you may adjust the frequency display of the spectrum analyzer to the value equal to the value of the carrier frequency.

6) Repeat part 5 for DSB-LC. Furthermore, using the spectrum analyzer, determine the modulation index, m.

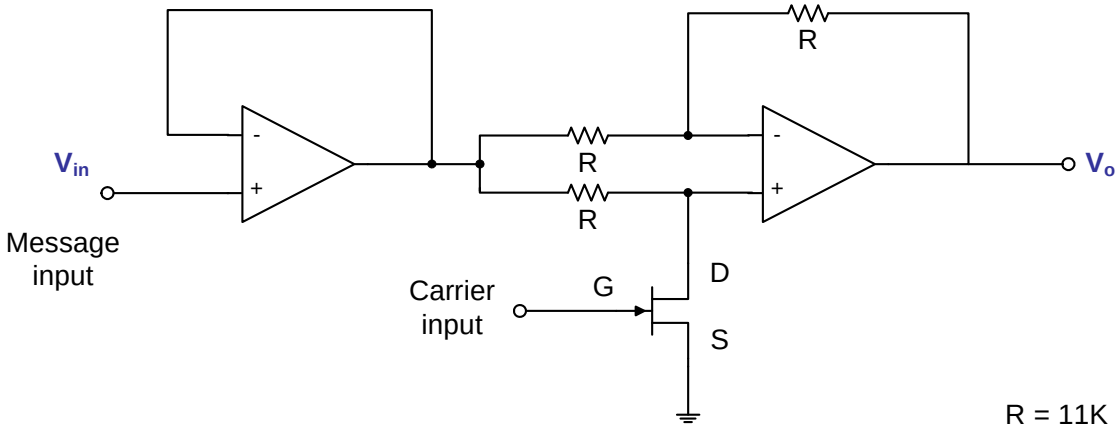


Figure 4.25 AM Modulation

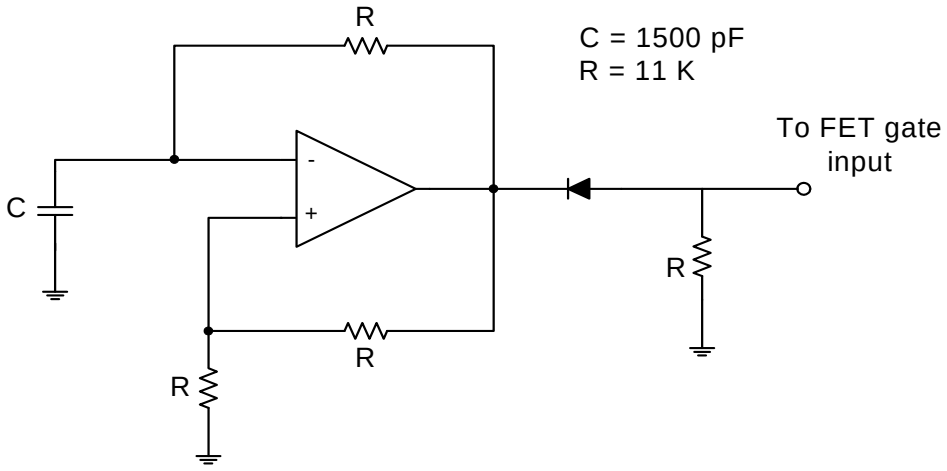


Figure 4.26 Carrier generator

6.2 Amplitude Demodulation

The overall general communication system block diagram is given in Figure 4.27. You have already built the modulator. As a channel, use a small length of wire. There are two types of AM demodulator: the envelop detector (no carrier waveform is required), and the synchronous demodulator (carrier waveform is needed).

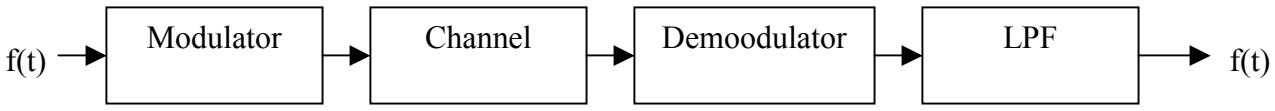


Figure 4.27 AM Block Diagram

6.3 Envelope detection

1) Connect the envelope detector of Figure 4.28.

2) Adjust the oscillator for a DSB-LC modulated waveform using sinusoidal modulation with $f_m = 100$ Hz. Connect $V_d(t)$ to the oscilloscope input.

3) Set C to the minimum value of the capacitor substitution box, noting the output waveform as C is increased. Sketch the optimum demodulated waveform. Compare $V_d(t)$ with the modulating waveform $V_{in}(t)$. Comment.

4) Repeat part 3) for a triangular modulating signal.

5) Now adjust for a DSB-SC waveform, comparing $V_d(t)$ with the modulating waveform $V_{in}(t)$. Does the envelope detector demodulate DSB-SC?

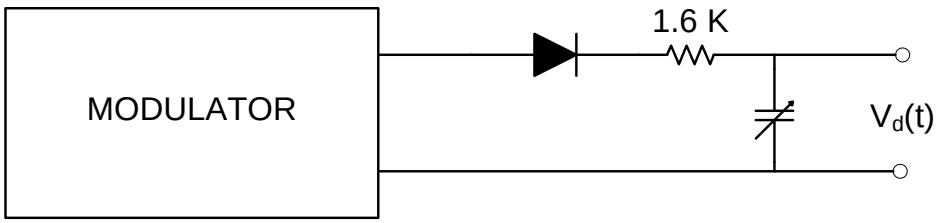


Figure 4.28 Envelope detector

6.4 Synchronous detection

1) Connect the synchronous detector of Figure 4.29.

2) Adjust for a DSB-LC signal. Compare $V_{in}(t)$ with $V_d(t)$. Sketch the waveforms. Has the message been recovered?

3) Adjust for a DSB-SC signal. Again compare $V_{in}(t)$ with $V_d(t)$. Sketch the waveforms. Is it possible to retrieve the original message? Explain.

4) Is it possible to demodulate a DSB-LC signal? A DSB-SC signal?

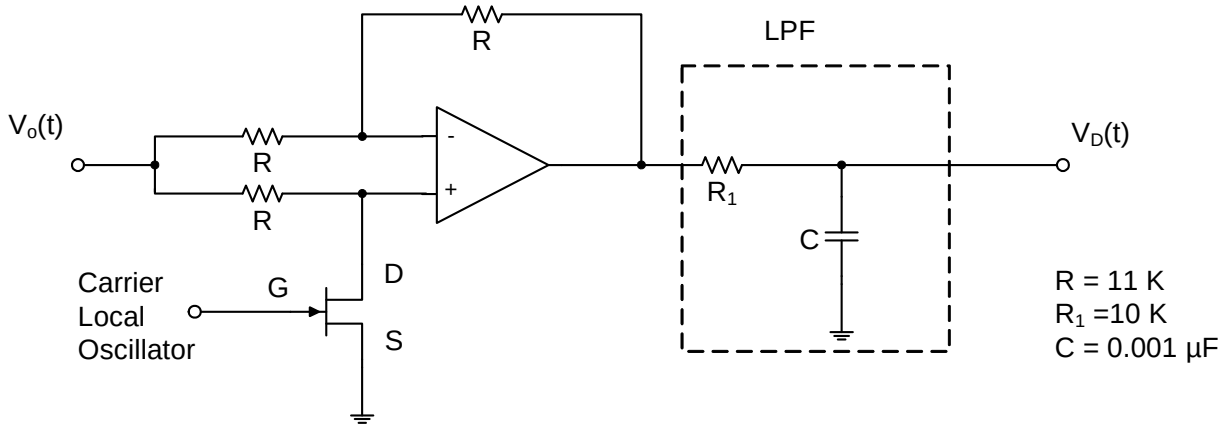


Figure 4.29 Synchronous Demodulation

7 Questions and Calculations

1. Why must m be less than 100% for envelope detection?

2. Compare Envelope Detection to Synchronous Detection and list at least one advantage and one disadvantage of each.

TABLE 4.1 Some Selected Fourier Transform Pairs

$f(t)$	$F(\omega) = \mathcal{F}\{f(t)\}$
1. $e^{-at}u(t)$	$1/(a + j\omega)$
2. $te^{-at}u(t)$	$1/(a + j\omega)^2$
3. $e^{-a\|t\|}$	$2a/(a^2 + \omega^2)$
4. $e^{-t^2/(2\sigma^2)}$	$\sigma\sqrt{2\pi}\, e^{-\sigma^2\omega^2/2}$
5. $\text{sgn}(t)$	$2/(j\omega)$
6. $j/(\pi t)$	$\text{sgn}(\omega)$
7. $u(t)$	$\pi\delta(\omega) + 1/(j\omega)$
8. $\delta(t)$	1
9. 1	$2\pi\delta(\omega)$
10. $e^{\pm j\omega_0 t}$	$2\pi\delta(\omega \mp \omega_0)$
11. $\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
12. $\sin\omega_0 t$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
13. $rect(t/\tau)$	$\tau Sa(\omega\tau/2)$
14. $\frac{W}{2\pi} Sa(Wt/2)$	$rect(\omega/W)$
15. $\frac{W}{\pi} Sa(Wt)$	$rect(\omega/2W)$
16. $\Lambda(t/\tau)$	$\tau[Sa(\omega\tau/2)]^2$
17. $\frac{W}{2\pi}[Sa(Wt/2)]^2$	$\Lambda(\omega/W)$
18. $\cos(\pi t/\tau)rect(t/\tau)$	$\frac{2\tau}{\pi}\frac{\cos(\omega\tau/2)}{1-(\omega\tau/\pi)^2}$
19. $\frac{2W}{\pi^2}\frac{\cos(Wt)}{1-(2Wt/\pi)^2}$	$\cos[\pi\omega/(2W)]rect[\omega/(2W)]$
20. $\delta_T(t)$	$\omega_0\delta_{\omega_0}(\omega)$ where $\omega_0 = 2\pi/T$

TABLE 4.2 Fourier Transforms Properties

Operation	$f(t) \quad \leftrightarrow$	$F(\omega)$
Linearity (Superposition)	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
Complex conjugate	$f^*(t)$	$F^*(-\omega)$
Scaling	$f(\alpha t)$	$\frac{1}{\lvert\alpha\rvert} F\left(\frac{\omega}{\alpha}\right)$
Delay	$f(t - t_0)$	$e^{-j\omega t_0} F(\omega)$
Frequency translation	$e^{j\omega_0 t} f(t)$	$F(\omega - \omega_0)$
Amplitude modulation	$f(t)\cos\omega_0 t$	$\frac{1}{2} F(\omega + \omega_0) + \frac{1}{2} F(\omega - \omega_0)$
Time convolution	$\int_{-\infty}^{\infty} f_1(\tau) f_2(t - \tau) d\tau$	$F_1(\omega) F_2(\omega)$
Frequency convolution	$f_1(t) f_2(t)$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(u) F_2(\omega - u) du$
Duality: time-frequency	$F(t)$	$2\pi f(-\omega)$
Symmetry: even-odd	$f_e(t)$	$F_e(\omega)$ [real]
	$f_o(t)$	$F_o(\omega)$ [imaginary]
Time differentiation	$\frac{d}{dt} f(t)$	$j\omega F(\omega)$
Time integration	$\int_{-\infty}^{t} f(\tau) d\tau$	$\frac{1}{j\omega} F(\omega) + \pi F(0)\delta(\omega)$, where $F(0) = \int_{-\infty}^{+\infty} f(t) dt$

Table 4.3 Characteristics of Modulation

MODULATION	CHARACTERISTICS
1. AMPLITUDE (AM)	• Disregarding noise, original signal can be reproduced exactly at the receiver. • Very susceptible to noise of all kinds. • Can transmit and decode wide band width signals
2. FREQUENCY (FM)	• Disregarding noise, original signal can be reproduced exactly at receiver. • Not very susceptible to noise. • Limited bandwidth
3. PULSE WIDTH (PWM)	• Similar to fm except some accuracy in reproducing the message is lost. • Easy to demodulate (LPF)
4. PULSE AMPLITUDE (PAM)	• Similar to AM except some accuracy is lost. • Can be used in multiplexing
5. PULSE CODED (PCM)	• Only discrete voltage levels can be encoded. • Repeaters can be used to reduce noise in long distance transmission since the code is binary. • Inefficient use of time because it requires synchronization bits. • Can handle fast slew rates.
6. DELTA (Δm)	• Does not need synchronization. • Simple, 2-bit code. • Cannot accurately encode fast slew rates
7. DELTA SIGMA(Δ-Σm)	• Same as delta except it can handle fast slew rates.

Laboratory 5
Frequency Modulation

1 Objective

To understand the principles of frequency modulation and demodulation.

2 Theory

2.1 Introduction

A sinusoidal carrier $c(t) = A\cos(\omega_c t + \theta_0)$ has three parameters that can be modified (modulated) according to an information signal $f(t)$.

1. Its amplitude *A*, which leads us to the class of systems designated as *amplitude modulating* (AM) systems.
2. Its frequency ω_c, which leads us to a class of systems designated as *frequency modulating* (FM) systems.
3. Its phase θ_0, which leads us to a class of systems designated as *phase modulating* (PM) systems.

We have already discussed the class of AM systems. In the sequel we focus on the class of FM and PM systems. Note that we can write that

$$c(t) = A\cos(\omega_c t + \theta_0) = A\cos(\theta(t)) \tag{5.1}$$

where $\theta(t)$ is often called the angle of the sinusoid. That's why FM and PM systems are sometimes referred to as *angle modulating* systems.

2.2 Preliminary notions of FM and PM Systems

Consider the carrier $c(t) = A\cos(\omega_c t + \theta_0)$. We can write

$$c(t) = A\cos(\theta(t)) \tag{5.2}$$

where we call $\theta(t)$ the *instantaneous phase* of the carrier. If we differentiate $\theta(t)$ with respect to time *t* we get a time function that we designate by $\omega(t)$ and we call it the *instantaneous frequency*. That is

$$\omega(t) = \frac{d\theta}{dt} \tag{5.3}$$

It is easy to see that the above definition of instantaneous frequency makes sense if we apply it to a pure carrier $c(t) = A\cos(\omega_c t + \theta_0)$, because then we get

$$\omega(t) = \omega_c \tag{5.4}$$

From equation (5.3), above, we see that if you have the instantaneous phase of a sinusoid, you can compute its instantaneous frequency by differentiation. Furthermore, if you know the instantaneous frequency $\omega(t)$ of a sinusoid you can compute its instantaneous phase by integration as follows:

$$\theta(t) = \int_0^t \omega(\tau) d\tau + \theta_0 \tag{5.5}$$

Obviously if we start with $\omega(t) = \omega_c$ we get

$$\theta(t) = \omega_c t + \theta_0 \tag{5.6}$$

Phase and frequency modulations are techniques that modify the instantaneous phase and frequency, respectively, of a sinusoid in a way dictated by an information signal $f(t)$.

2.3 Phase Modulation

Here the information signal $f(t)$ is placed as a linear term in the instantaneous phase of the carrier. That is

$$\theta(t) = \omega_c(t) + \theta_0 + k_p f(t) \tag{5.7}$$

where k_p is a constant of the modulating device. Hence, the PM modulated signal is equal to

$$m_p(t) = A\cos(\omega_c t + \theta_0 + k_p f(t)) \tag{5.8}$$

2.4 Frequency Modulation

Here the information signal gets inserted as a linear term into the instantaneous frequency of the carrier. That is,

$$\omega(t) = \omega_c + k_f f(t) \tag{5.9}$$

where k_f is a constant due to the modulator. In this case the instantaneous phase is equal to

$$\theta(t) = \omega_c t + \theta_0 + k_f \int_0^t f(\tau) d\tau \tag{5.10}$$

and as a result, the FM modulated signal looks like

$$m_f(t) = A\cos\left(\omega_c t + \theta_0 + k_f \int_0^t f(u) du\right) \tag{5.11}$$

A plot of FM and PM signals is shown in Figure 5.1.

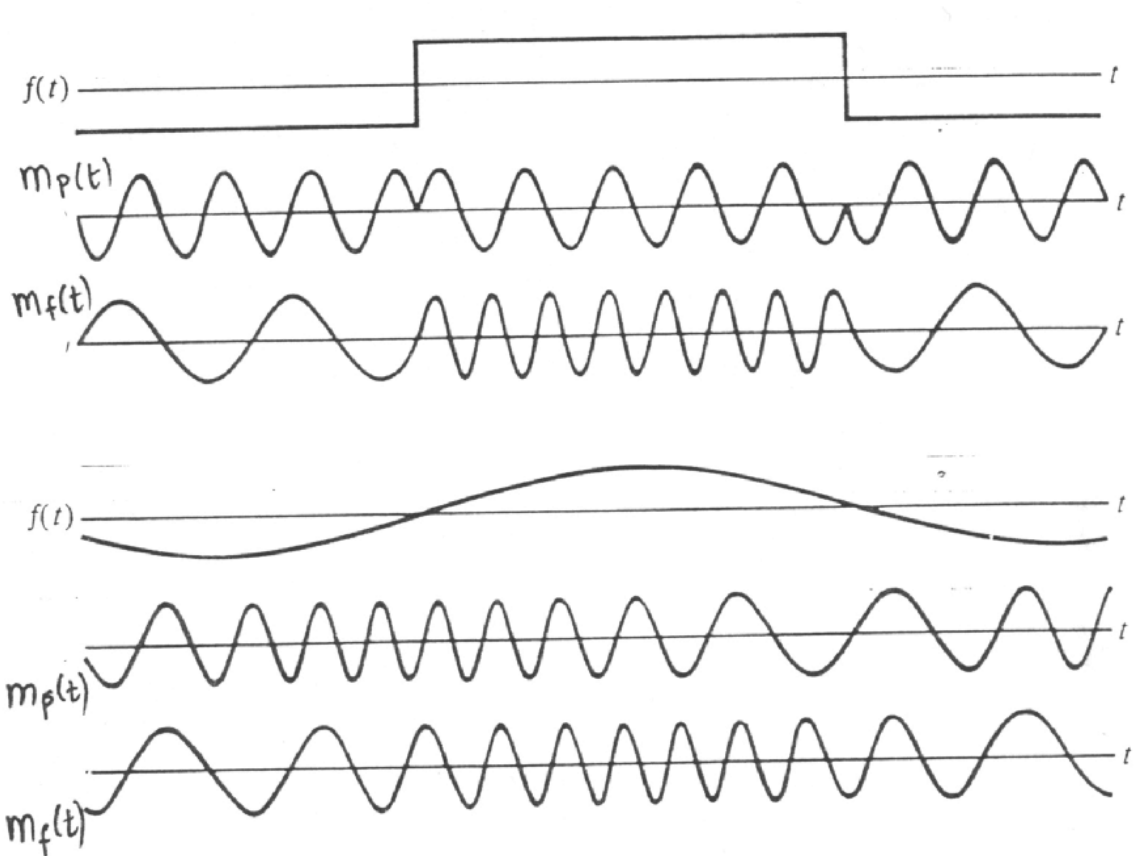


Figure 5.1 Examples of frequency and phase modulation

Now let us discuss FM and PM simultaneously and get a better insight into their similarities and differences. For simplicity assume that $\theta_0 = 0$. Then,

$$m_p(t) = A\cos\left(\omega_c t + k_p f(t)\right) \tag{5.12}$$

and

$$m_f(t) = A\cos\left(\omega_c t + k_f \int_0^t f(u)du\right) \tag{5.13}$$

Let us take $m_p(t)$ and find its instantaneous frequency $\omega(t)$. Indeed,

$$\omega(t) = \omega_c + k_p \frac{df(t)}{dt} \tag{5.14}$$

The above equation tells us that in the PM case the instantaneous frequency has a linear term proportional to the derivative of the information signal $f(t)$. So, we can say that the PM case is in reality the FM case but with an information signal being the derivative of the actual information signal. In other words, if we have a device that produces FM signals we can make it to produce PM signals by giving it as an input the derivative of the information signal. The above equation also tells us that the FM case is in reality a PM case with the modulating signal being the integral of the information signal. Also, if we have a device that produces PM, we can make it to produce FM by providing to it as an input the integral of the information signal.

Hence, what it boils down to is that we need to discuss either PM or FM and not both. We choose to focus on FM, which is used to transmit baseband analog signals, such as speech or music. PM is primarily used in transmitting digital signals.

Our primary focus in the examination of FM signals will be the analysis of its frequency characteristics. Although it has been a straightforward task to find the Fourier transform of an AM signal the same is not true for FM signals. Let us again consider the general form of an FM signal

$$m_f(t) = A\cos(\omega_c t)\cos\left(k_f g(t)\right) - A\sin(\omega_c t)\sin\left(k_f g(t)\right) \tag{5.15}$$

Where $g(t) = \int_0^t f(u)du$. If we are able to find the FT of $\cos\left(k_f g(t)\right)$ and $\sin\left(k_f g(t)\right)$ we can produce the FT of the signal $m_f(t)$ without a lot of effort. The FT of $\cos\left(k_f g(t)\right)$ and $\sin\left(k_f g(t)\right)$ cannot be found for any $g(t)$ and in fact it has been found for few $f(t)$'s. To get a deeper insight consider $\cos\left(k_f g(t)\right)$ and expand it in terms of its Taylor series.

$$\cos\left(k_f g(t)\right) = 1 - \frac{k_f^2 g^2(t)}{2!} + \frac{k_f^4 g^4(t)}{4!} - \frac{k_f^6 g^6(t)}{6!} + \cdots \tag{5.16}$$

If the signal $f(t)$ is known, its Fourier transform $F(\omega)$ is also known, and the Fourier transform of $g(t)$ can be computed. In particular, from well known Fourier transform properties we can deduce that

$$\begin{aligned} g^2(t) &\rightarrow G(\omega) * G(\omega) \\ g^4(t) &\rightarrow G(\omega) * G(\omega) * G(\omega) * G(\omega) \\ \vdots \quad &\rightarrow \vdots \end{aligned} \tag{5.17}$$

Based on the above two equations (5.16) and (5.17) we can state that to compute the Fourier transform of $\cos(k_f g(t))$ for an arbitrary signal $g(t)$ becomes a formidable task (we need to compute a lot of convolutions). Furthermore, it seems that the bandwidth of $\cos(k_f g(t))$ is infinite (note that every time we multiply a signal with itself in the time-domain we double its bandwidth in the frequency domain). Hence if the bandwidth of $g(t)$ is ω_m, the bandwidth of $g^2(t)$ is $2\omega_m$, the bandwidth $g^4(t)$ is $4\omega_m$, and so on. In reality though not all the terms in the Taylor series expansion of $\cos(k_f g(t))$ contribute equally to the determination of the signal $\cos(k_f g(t))$. Notice that in the Taylor series expansion of $\cos(k_f g(t))$ the coefficients multiplying the powers of $g(t)$ get smaller and smaller. This observation will leads us into the conclusion that the bandwidth of the terms $\cos(k_f g(t))$ and $\sin(k_f g(t))$is indeed finite, and as a result the bandwidth of the FM signal $m_f(t)$ is also finite.

To investigate the bandwidth of an FM signal thoroughly we will discriminate two cases of FM signals: The case of *Narrowband FM* and the case of *Wideband FM.*

2.6 Narrowband FM

Consider again the FM signal $m_f(t)$ given by the following equation.

$$m_f(t) = A\cos(\omega_c t)\cos(k_f g(t)) - A\sin(\omega_c t)\sin(k_f g(t)) \tag{5.18}$$

The terms for which FT is difficult to evaluate are: $\cos k_f g(t)$ and $\sin k_f g(t)$. Each one of these terms has a Taylor series expansion involving infinitely many terms. Let us see what happens if each one of these terms is approximated only by their first term in the Taylor series expansion. Then,

$$\begin{aligned} \cos k_f g(t) &\approx 1 \\ \sin k_f g(t) &\approx k_f g(t) \end{aligned} \tag{5.19}$$

Obviously if we make the above substitutions in equation (5.18) we get

$$m_f(t) = A\cos(\omega_c t) - Ak_f g(t)\sin(\omega_c t) \tag{5.20}$$

The advantage of the above equation is that we can evaluate its FT, and consequently the FT of $m_f(t)$. It is not difficult to see that the bandwidth of $m_f(t)$ is approximately equal to 2 times the bandwidth of $f(t)$ ($f(t)$ is the information signal). Hence, when the above approximations are accurate we are generating an FM signal whose bandwidth is approximately equal to the bandwidth of an AM signal. Since, in most cases an FM signal will occupy much more bandwidth than an AM signal, the aforementioned type of FM signal is called *narrowband FM.* In the sequel, we are going to identify (quantitatively) conditions under which we can call an FM signal narrowband. These conditions will be a byproduct of our discussion of wideband FM systems.

2.7 Wideband FM

To illustrate the ideas of wideband FM let us start with the simplest of cases where the information signal is a single sinusoid. That is,

$$f(t) = a\cos(\omega_m t) \tag{5.21}$$

Then, the instantaneous frequency of your FM signal takes the form:

$$\omega(t) = \omega_c + k_f a\cos(\omega_m t) \tag{5.22}$$

Integrating the instantaneous frequency $\omega(t)$ we obtain the instantaneous phase $\theta(t)$*:*

$$\theta(t) = \omega_c t + \frac{k_f a}{\omega_m}\sin(\omega_m t) \tag{5.23}$$

Consequently, the FM modulated wave is

$$m_f(t) = A\cos\left(\omega_c t + \frac{k_f a}{\omega_m}\sin(\omega_m t)\right) \tag{5.24}$$

The quantity $k_f\, a/\omega_m$ is denoted by β and it is referred to as the *modulation index* of the FM system. Let us now write the above expression for the signal $m_f(t)$ in a more expanded form.

$$m_f(t) = A\cos(\omega_c t)\cos(\beta\sin\omega_m t) - A\sin(\omega_c t)\sin(\beta\sin\omega_m t) \tag{5.25}$$

To simplify our discussion, from now on, we will be referring to the quantity $\cos(\beta\sin(\omega_m t))$ as *term A* and to the quantity $\sin(\beta\sin(\omega_m t))$ as *term B.* Terms A and B are the real and the imaginary part of the following complex exponential function

$$e^{j\beta\sin\omega_m t} \tag{5.26}$$

which is a periodic function with period $\frac{2\pi}{\omega_m}$. The above function can be expanded as an exponential Fourier series, as follows:

$$e^{j\beta \sin \omega_m t} = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_m t} \tag{5.27}$$

where the coefficients C_n are calculated by the equation:

$$C_n = \frac{\omega_m}{2\pi} \int_{-\pi/\omega_m}^{\pi/\omega_m} e^{j(\beta \sin \omega t - n\omega t)} dt \tag{5.28}$$

Let us now make a substitution of variables in the above equation. In particular, let us substitute $\omega_m t$ with x. Then,

$$C_n = \frac{1}{2\pi} \int_{\pi}^{\pi} e^{j(\beta \sin x - nx)} dx \tag{5.29}$$

The above integral cannot be evaluated in closed form, but it has been extensively calculated for various values of β's and most n's of interest. It has a special name, called *the* nth *order Bessel function of the first kind* and argument β. This function is commonly denoted by the symbol $J_n(\beta)$. Therefore, we can write:

$$C_n = J_n(\beta) \tag{5.30}$$

As a result,

$$e^{j\beta \sin \omega_m t} = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{jn\omega_m t} \tag{5.31}$$

often called the *Bessel Jacobi equation.* If we evaluate the real and imaginary parts of the right hand side of equation (5.31) we will be able to calculate term A and term B, respectively. It turns out that if we substitute these values for term A and term B in the original equation for $m_f(t)$ (see equation (5.25)) we will end up with

$$m_f(t) = \sum_{-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m)t \tag{5.32}$$

The property of the Bessel function coefficients (actually Property 1) that led us to the above results is listed below. Some additional properties of the Bessel function coefficients are also listed.

1. $J_n(\beta)$ is real valued.

2. $J_n(\beta) = J_{-n}(\beta)$ for n even.

3. $J_n(\beta) = -J_{-n}(\beta)$ for n odd.

4. $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$

The advantage of equation (5.32) compared to the original equation (5.24) that defined the FM signal $m_f(t)$ is that now, through equation (5.32), we can compute the FT of the signal $m_f(t)$. It will consist of an infinite sequence of impulses located at positions $\omega_c + n\omega_m$, where n is an integer. In reality though, no matter how big β is, the significant $J_n(\beta)$'s will be only for indices $n \le \beta + 1$. Hence, the approximate bandwidth of your FM signal, when the information signal is of sinusoidal nature, is given by the following equation.

$$B = 2(\beta + 1)\omega_m \tag{5.33}$$

In Figure 5.2 various plots of the Bessel function coefficients $J_n(\beta)$ are shown. As we can see these plots verify our claim, above, that $J_n(\beta)$ become small for indices $n > \beta + 1$. In Figures 5.3 we show the FT of signals $m_f(t)$ for various β values.

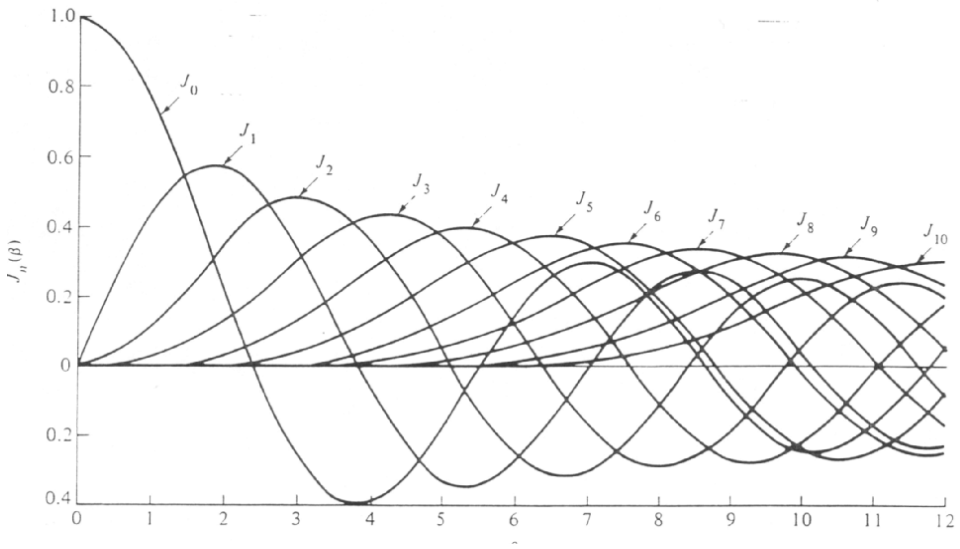


Figure 5.2 Plot of Bessel function of the first kind, $J_n(\beta)$

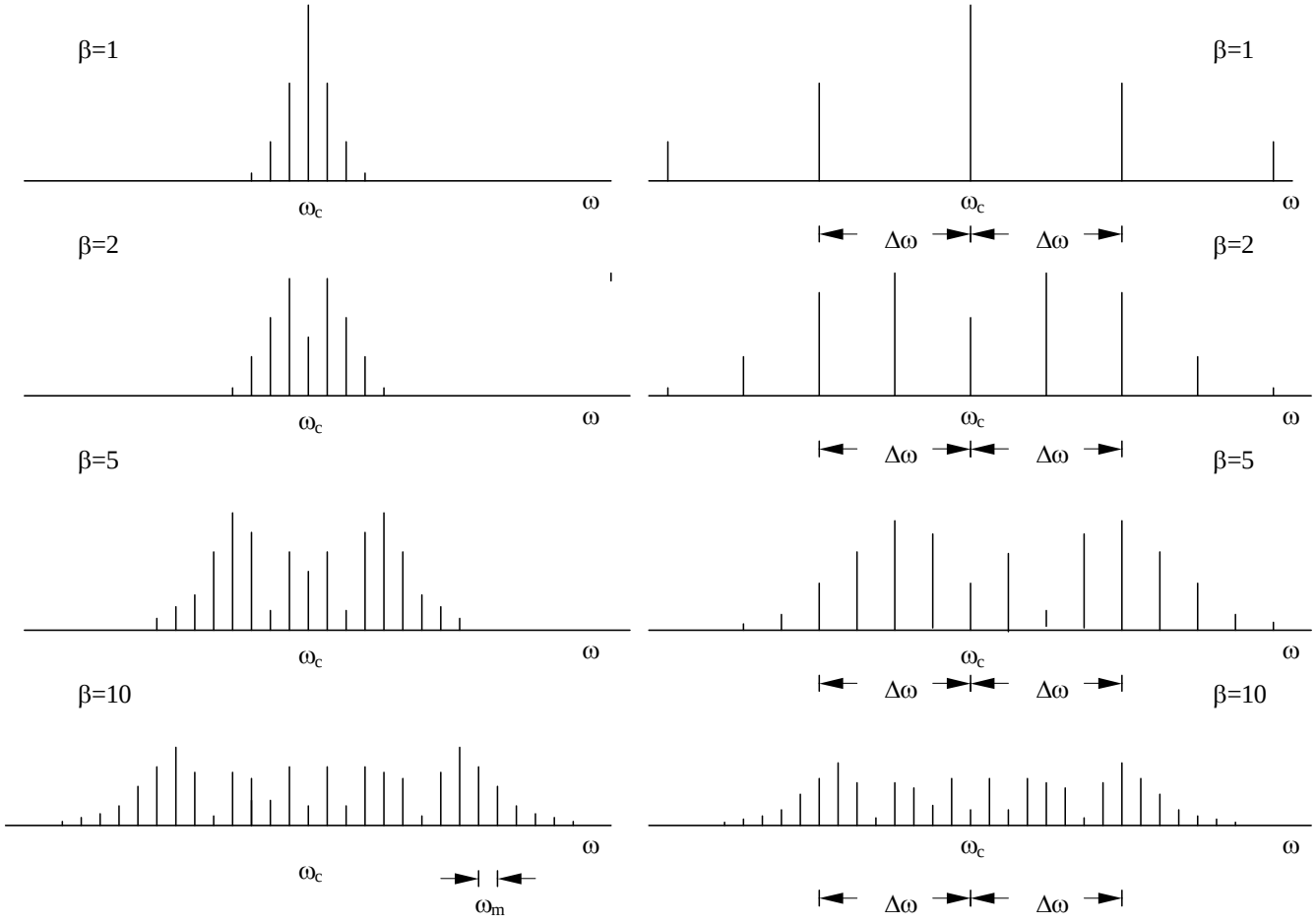


Figure 5.3: Magnitude line spectra for FM waveforms with sinusoidal modulation (a) for constant ω_m; (b) for constant $\Delta\omega$

In Table 5.1 the values of the Bessel function coefficients $J_n(\beta)$ are shown for various β values. We can use the values of the Table 5.1 to evaluate the bandwidth of the signal $m_f(t)$ as follows. We are still operating under the assumption that the information signal is of sinusoidal nature. As a result, expression (5.32) is a valid representation of our signal $m_f(t)$. Let us now impose the criterion that for the evaluation of the bandwidth of the signal $m_f(t)$ we are going to exclude all terms of the infinite sum with index $n_{\max}$, such that $|J_n(\beta)| < 0.01$ for $n > n_{\max}$. This criterion is often called the 1% *criterion* for the evaluation of bandwidth. If we find that $n_{\max}$ is the minimum index n that does not violate the 1% criterion then we can claim that the approximate bandwidth of our signal, according to the 1% criterion, is:

$$B = 2n_{\max}\omega_m \tag{5.34}$$

It is worth mentioning that the evaluation of bandwidth based on equation (5.33) corresponds to the bandwidth of your FM signal according to a 10% criterion.

The above procedure followed for the evaluation of the FT of $m_f(t)$, can be extended to the cases where the information signal $f(t)$ is a sum of sinusoidal signals, or a periodic signal. In particular, if

$$f(t) = a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t) \tag{5.35}$$

then the phase of our FM signal $m_f(t)$ is provided by the following equation.

$$\theta(t) = \omega_c t + \frac{k_f a_1}{\omega_1} \sin(\omega_1 t) + \frac{k_f a_2}{\omega_2} \sin(\omega_2 t) \tag{5.36}$$

Omitting the details, we arrive at a representation of the signal $m_f(t)$ such that

$$m_f(t) = A \sum_{n=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} J_n(\beta_1) J_k(\beta_2) \cos(\omega_c + n\omega_1 + k\omega_2)t \tag{5.37}$$

where $\beta_1 = a_1 k_f / \omega_1$ and $\beta_2 = a_2 k_f / \omega_2$. As we can see from the above equation, we now have impulses at $\omega_c \pm n\omega_1, \omega_c \pm k\omega_2$ as well as $\omega_c \pm n\omega_1 \pm k\omega_2$.

Most of the aforementioned discussion regarding the FT and the bandwidth of an FM signal $m_f(t)$ is based on the assumption that the information signal $f(t)$ is a sinusoid, a sum of sinusoids, or a periodic signal. We want to be able to derive a formula for the bandwidth of an FM signal $m_f(t)$ for an arbitrary information signal $f(t)$. Let us revisit the approximate FM signal bandwidth formula (5.33) derived for a sinusoidal information signal

$$B = 2(\beta + 1)\omega_m = 2(ak_f + \omega_m) \tag{5.38}$$

The second equality in (5.38) is obtained by substituting the value of β with its equal. Now let us pay a closer look at the two terms involved in the evaluation of the approximate bandwidth B. The first term ak_f is the maximum frequency deviation of the instantaneous frequency $\omega(t)$ from the carrier frequency ω_c; it is often denoted by $\Delta\omega$. The second term ω_m is the maximum frequency content of the information signal $f(t)$. Keeping these two clarifications in mind, we now define the approximate bandwidth of an FM signal $f(t)$ to be equal to

$$B = 2(\Delta\omega + \omega_m) \tag{5.39}$$

where $\Delta\omega$ is the maximum frequency deviation from the carrier frequency, and ω_m is the maximum frequency content of the information signal $f(t)$. It is not difficult to show that for an arbitrary information signal $f(t)$, $\Delta\omega = k_f \max_t |f(t)|$. Furthermore, to find ω_m we first need to compute the FT of the information signal $f(t)$. Hence, based on the above equation we can claim that the approximate bandwidth of an FM signal is computable even for the case of an arbitrary info signal. It is worth pointing out that the bandwidth formula given above has not been proven to be true for FM signals produced by arbitrary info signals; but it has been verified experimentally in a variety of cases. Equation (5.39) is referred to as *Carson's formula (rule)* for the evaluation of the bandwidth of an FM signal, and from this point on it can be applied freely, independently of whether the information signal is of sinusoidal nature or not.

One of the ramifications of Carson's rule is that we can increase the bandwidth of an FM signal at will, by increasing the modulation constant k_f, or equivalently, by increasing the peak frequency deviation $\Delta\omega$. One of the advantages of increasing the bandwidth of the FM signal is that larger bandwidths result in FM signals that exhibit better tolerances to noise. Unfortunately the peak frequency deviation of an FM signal is constrained by other considerations, such as a limited overall bandwidth that needs to be shared by a multitude of FM users. For example, in commercial FM the peak frequency deviation is specified to be equal to 75 KHz.

Let us now say a word about PM. From our previous discussions the form of a PM signal produced by a sinusoidal modulating signal is as follows:

$$m_p(t) = A\cos\left(\omega_c t + k_p a\cos(\omega_m t)\right) \tag{5.40}$$

and the instantaneous frequency is equal to

$$\omega(t) = \omega_c - k_p a\omega_m \sin\omega_m t \tag{5.41}$$

Hence,

$$\Delta\omega = ak_p\omega_m \tag{5.42}$$

That is $\Delta\omega$ depends on ω_m. This is considered a disadvantage compared to commercial FM, where $\Delta\omega$ is fixed. Carson's rule is also applicable for PM systems but to find the peak frequency deviation of a PM system you need to find the maximum, with respect to time, of $\left|\dot{f}(t)\right|$. where $\dot{f}(t)$ is the time derivative of $f(t)$. Actually, for PM systems

$$\Delta\omega = k_p \max\left|\dot{f}(t)\right| \tag{5.43}$$

One last comment to conclude our discussion of angle modulation systems. It can be shown that the power of an FM or PM signal of the form

$$m(t) = A\cos(\theta(t)) \tag{5.44}$$

is equal to $A^2/2$

3 Pre-lab Questions

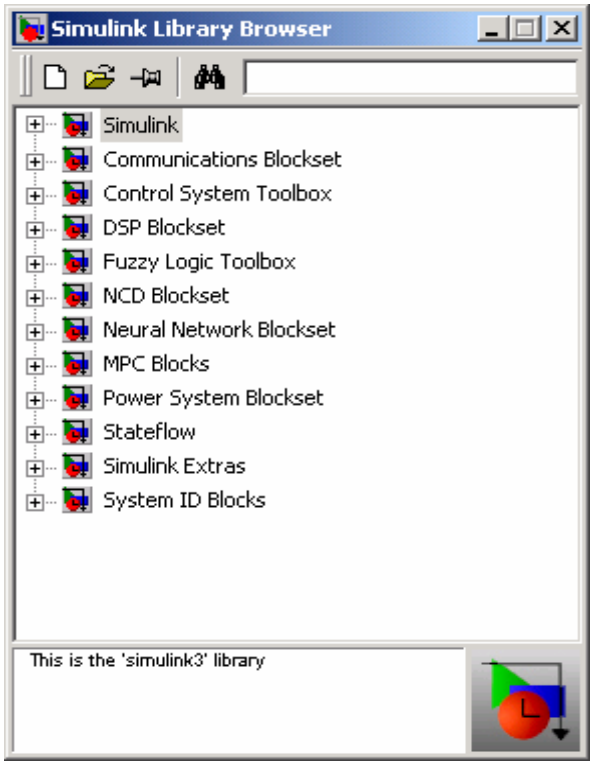
1 Provide a paragraph, where you compare AM and FM modulation (advantages, disadvantages)

2 Why is the use of FM more preferred than PM? Explain your answer. (Hint: Compare the frequency deviations in both cases)

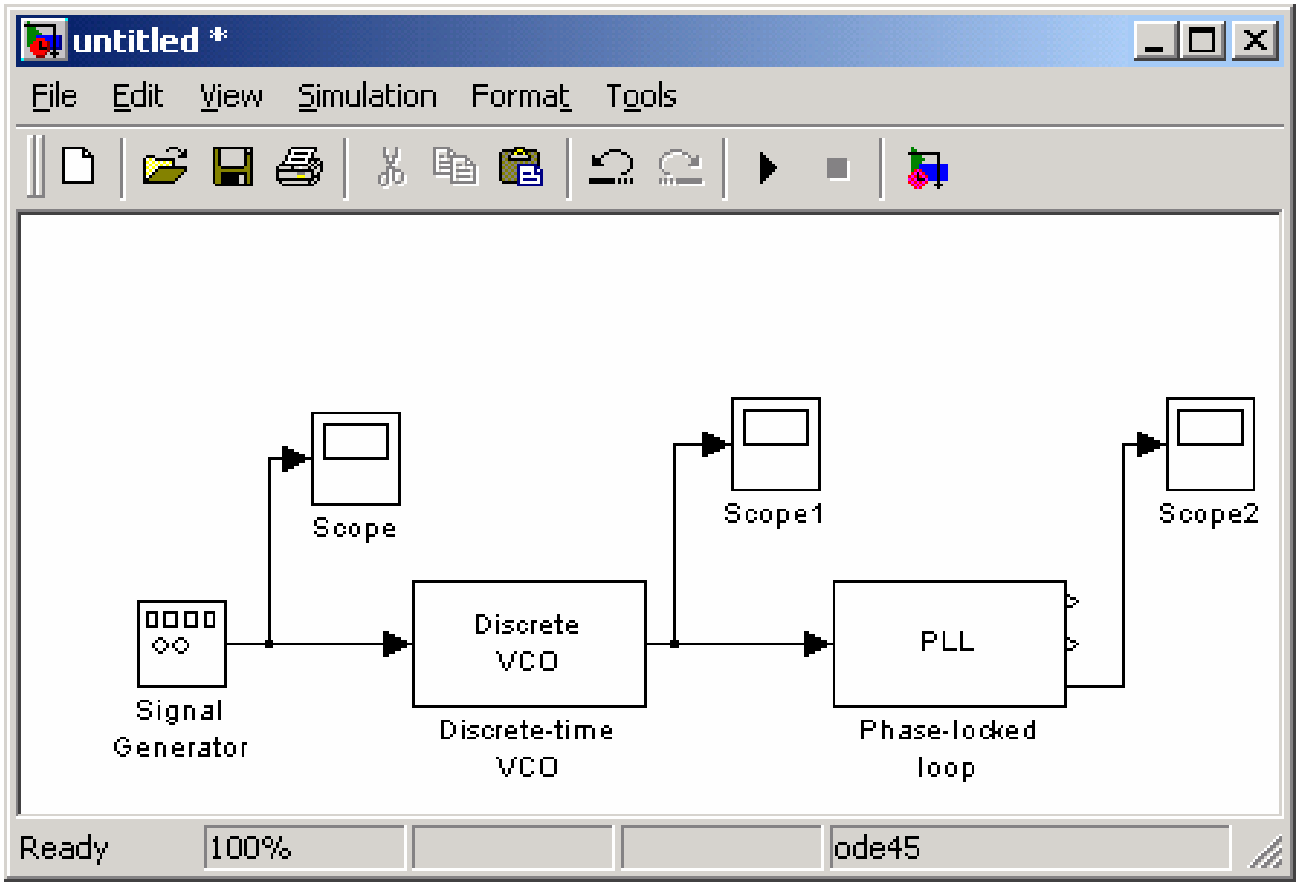
3 Give a short qualitative justification of the fact that FM is more noise immune than AM.

4 Simulation

We are going to do the simulation in Simulink of Matlab. To run the Simulink, enter the 'simulink' command in the Matlab command window. It should look as it is shown in the following. Open a new window using the left icon.



Construct the following block diagram for the FM modulation and demodulation simulation. The following blocks will be necessary for your simulation. The paths of the blocks have been given as well.



Block	Path
Signal Generator	Simulink -> Source
Discrete time VCO	Communications Block set -> Comm Sources->Controlled Sources
Phase Lock Loop	Communications Block set -> Synchronization
Scope	Simulink -> Sink

Just drag and drop the blocks you need for your simulation in your work window. Click the left mouse button, hold and drag to connect the blocks by wire. The parameters for the individual module have been given as an example. To set the parameters of a block just double click on it.

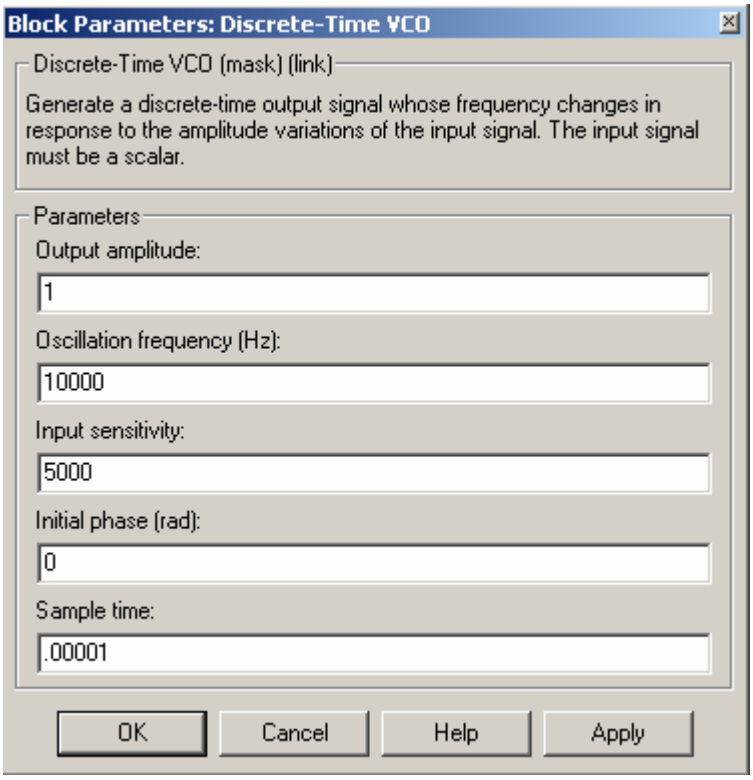
- ***Signal Generator***

  Wave form: sine

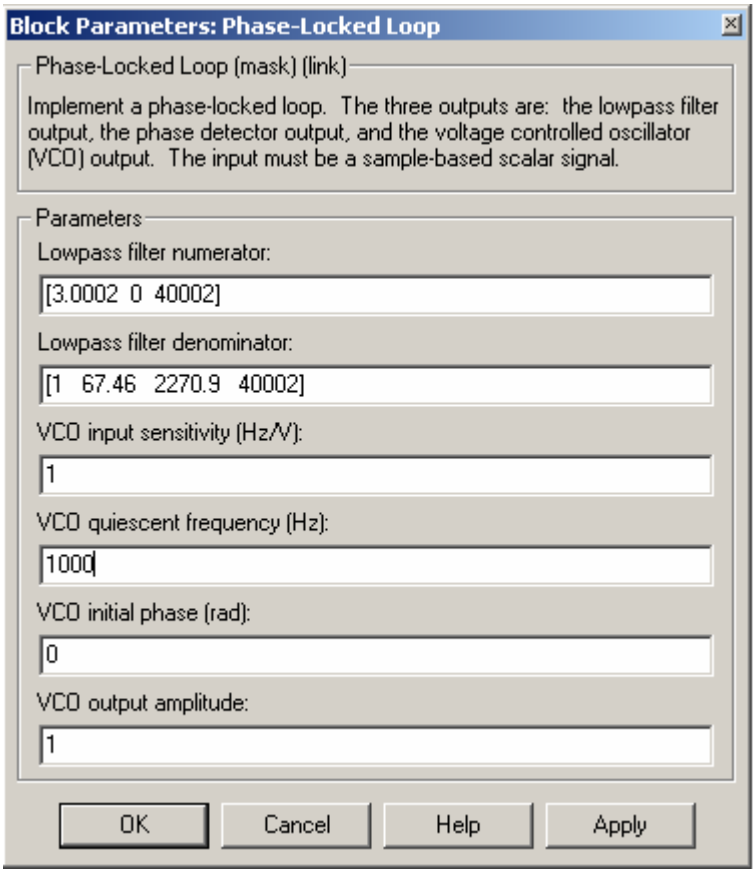
  Amplitude: 1

  Frequency: 1000

- ***Discrete time VCO***



♦ ***Phase Lock Loop***



Set the simulation stop time at say 0.002 from the simulation -> parameters menu.

Run the simulation from the simulation -> start menu. Then double click on the scopes to see the time domain signals.

Do the simulation with square wave input signal. Is the demodulated signal the faithful reproduction of the input signal? Why?

5 Implementation

In linear FM (frequency modulation), the instantaneous frequency of the output is linearly dependent on the voltage at the input. Zero volts at the input wil1 yield a sinusoid at center frequency f_c at the output. The equation for a FM signal is:

$$m_f(t) = \cos(\omega_c t + g(t)) \tag{5.45}$$

Where $\omega_c t + g(t) = \phi_f(t)$(instantaneous phase) and the instantaneous angular frequency is:

$$\omega_f(t) = \omega_c + \frac{dg(t)}{dt} \tag{5.46}$$

In linear FM the instantaneous frequency can be approximated by a straight line.

$$\omega_f(t) = \omega_c + k_f f(t) \tag{5.47}$$

Where k_f is a constant and $f(t)$ is the input signal.

$$\phi_i(t) = \int (\omega_c + k_f f(t))dt = \omega_c t + 2\pi k_f \int f(t)dt \tag{5.48}$$

$$m_f(t) = \cos\left(\omega_c t + 2\pi k_f \int f(t)dt\right) \tag{5.49}$$

Note that the amplitude of an FM signal never varies.

For a sinusoid input and positive k_f, the modulated FM waveform wil1 relate to the input as shown in figure 5.1.

Depending on the VCO, k_f can be positive or negative and f_c can also vary and is a function of external timing resistor and capacitor values. The relationship between input voltage and output frequency at any given point in time is:

$$f_{out} = f_c + k_f V_{in} \tag{5.50}$$

where k_f is positive for figure 5.1 and has units of Hz/Volt.

Let $f(t) = A\cos\omega_m t$, then

$$m_f(t) = \cos\left(\omega_c t + \frac{ak_f}{f_m}\sin\omega_m t\right) \tag{5.51}$$

The peak frequency deviation from ω_c is $ak_f 2\pi$ (radian/sec), and the total peak-to-peak deviation is $2(ak_f 2\pi)$. The modulation index β for this signal is:

$$\beta = \frac{ak_f}{f_m} \tag{5.52}$$

Note that β will vary for each frequency component of the signal.

5.1 Spectrum of FM

The spectrum of an FM signal is described by Bessel functions. As shown in section 2.7, for a single frequency, constant amplitude message, the spectrum of $m_f(t)$ is:

$$m_f(f) = \sum J_n(\beta)\cos(\omega_c t + n\omega_m t) \tag{5.53}$$

where $\frac{ak_f}{f_m} = \beta =$ FM modulation index and $J_n(\beta) = \frac{1}{2\pi}\int e^{j\beta\sin\beta - J_n(\beta)}d\theta$, which is the n^{th} order Bessel function eva1uated at β. Therefore, for each frequency at the input there are infinite number of spectral components at the output with the amplitude or each component determined by the modu1ation index β.

The amplitude of the higher order terms will decrease, on the average, such that there will be a limited bandwidth where most of the energy is concentrated.

5.2 VCO (FM modulator)

A voltage-controlled oscil1ator (VCO) converts the voltage at its input to a corresponding frequency at its output. This is accomplished by a variable reactor (varactor) where the reactance varies with the voltage across it. The varactor is part of a timing circuit, which sets the VCO output frequency. There are two limitations for most VCO's:

1. The input voltage must be small (usually there is an attenuating circuit at the input).

2. The bandwidth is limited for a linear frequency-to-voltage relationship.

5.3 PLL (FM demodulator)

The phase-locked loop (PLL) is used to demodulate FM. The phases of the input and feedback signals are compared and the PLL works to make the phase difference between the two signa1s equa1 to zero.
Figure 5.4 shows a basic block diagram of a PLL.

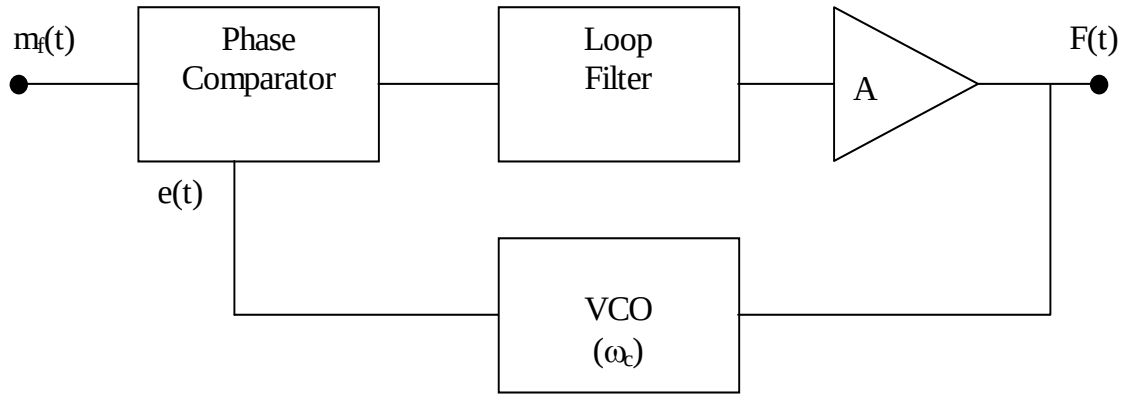


Figure 5.4. Phase Locked Loop (PLL) block diagram

The VCO in the feedback is an FM modu1ator, and the center frequency .can be set equal to the center frequency of $m_f(t)$.

The equation for $m_f(t)$ is

$$m_f(t) = \cos\left(\omega_c t + 2\pi k_f \int f(\alpha) d\alpha\right) \tag{5.54}$$

where $k_f \int f(\alpha) d\alpha = \theta_1 t$ and the equation of $e(t)$ is

$$e(t) = \cos\left(\omega_c t + k_d \int F(\alpha) d\alpha\right) \tag{5.55}$$

where $k_d \int f(\alpha) d\alpha = \theta_2 t$

Since each signal has the same center frequency ω_c, the phase comparator compares the instantaneous values of $\theta_1(t)$ and $\theta_2(t)$. The difference in phase is transformed into a DC voltage level proportiona1 to the phase difference and then amplified to yield $F(t)$. This voltage is the input for the VCO in the feedback. As a result, the difference between $\theta_1(t)$ and $\theta_2(t)$ will be made smaller. This process occurs continually, such that $\theta_1(t) = \theta_2(t)$ at all times.

Substituting for $\theta_1(t) = \theta_2(t)$

$$k_d \int F(\alpha) d\alpha = k_f \int f(\alpha) d\alpha \text{ and } F(t) = \frac{k_f}{k_d} f(t). \tag{5.56}$$

where $f(t)$ was the original message signal and $F(t)$ is the demodulated output.

6 Equipment

TDS5052B Oscilloscope
Function Generator
You should bring a floppy disk to store your waveforms.

7 Procedure

7.1 Modulation

a. Build the modulator as shown in Figure 5.5(a).

b. Determine the constant k_f from the following:

1) Use your triple output power supply to apply the input voltages specified in the following table and record the output frequency.

V_{in} (Volts)	f_{out} (KHz)
0.0	
+1.0	
+2.0	
+3.0	
+4.0	
+5.0	
+6.0	

2) Plot f_{out} vs V_{in}. Draw the best straight line through these points. The slope of this line is k_f. Note that k_f has units of Hz/Volts. What is the measured value of k_f?

(3) According to the XR2206 data sheet, the expected voltage-to-frequency conversion gain is $k_f = -\frac{0.32}{R_1 C_1} Hertz/Volts$.

Calculate the expected value of k_f. Assuming the resistors have a tolerance of ±10% and the capacitors have a tolerance of ±20%, how do the measured and calculated values compare?

c. Add input coupling capacitor C_2 to the circuit as shown in Figure 5.5 (b). Set the modulation frequency to f_m = 2 KHz. Fill in the following table using the following equations:

$$\Delta f_{peak} = \beta f_m = \alpha k_f \quad (5.57)$$

$$|V_{iv}| = \Delta f_{peak} / k_f \quad (5.58)$$

β (modulation index)	Calculated Δf_{peak} (Hz)	Calculated V_{in}	Measured Δf_{peak} (Hz)
4.5			
5.25			
9			
10.5			
12.3			

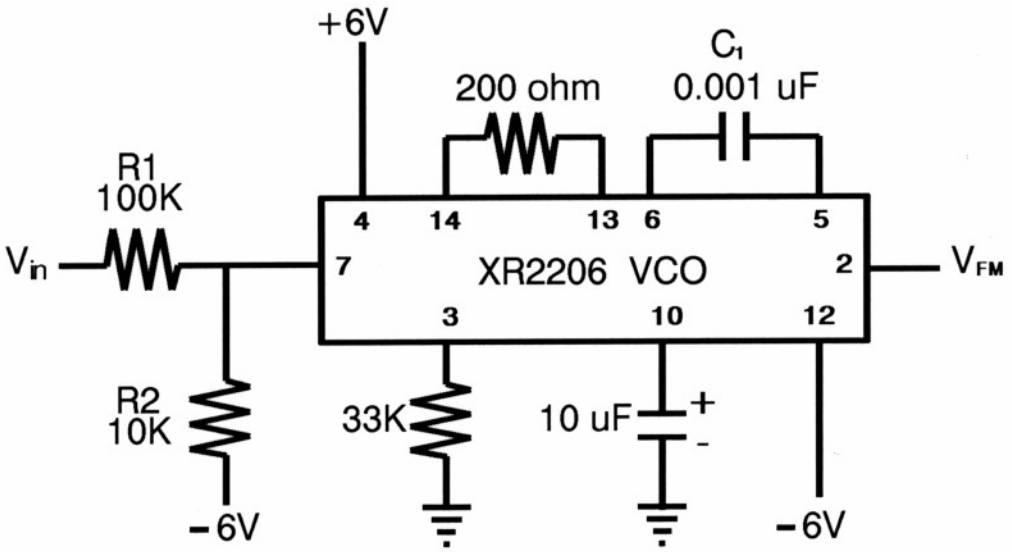


Figure 5.5 (a) VCO Determination of modulation constant K_f

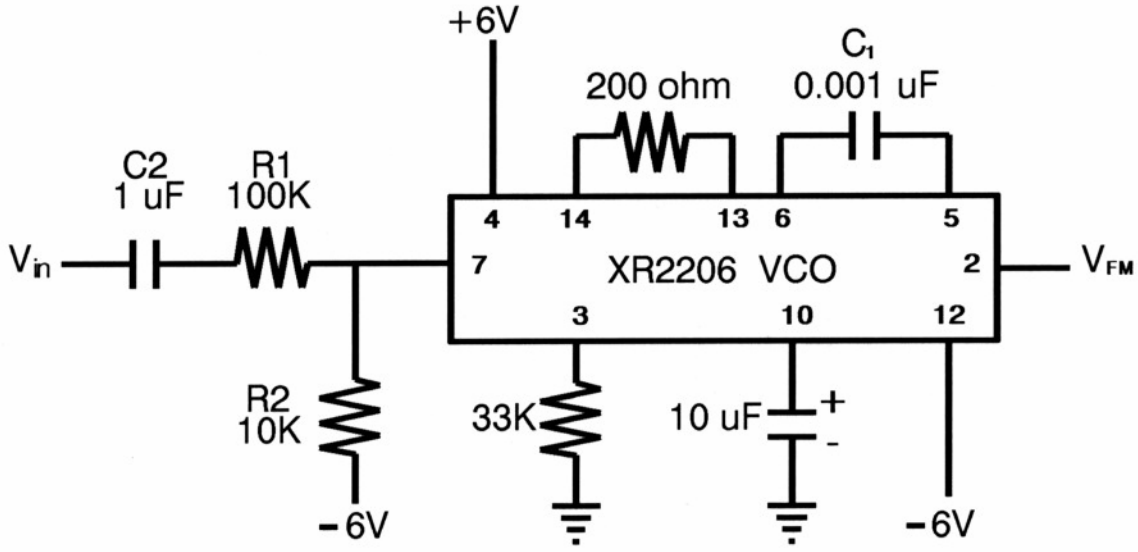
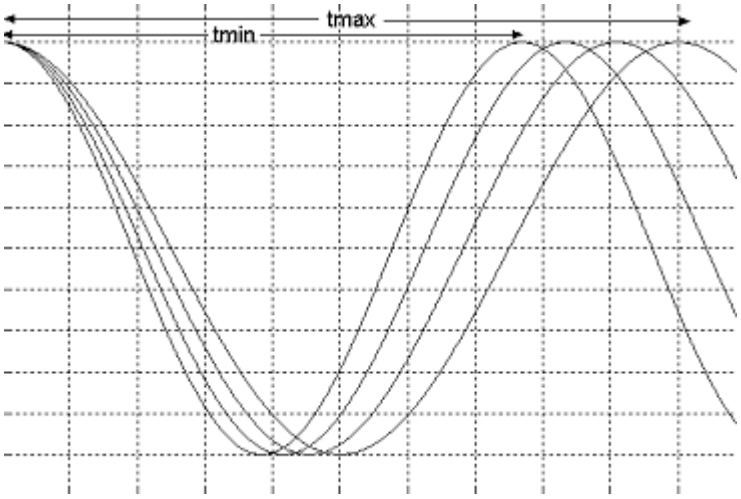


Figure 5.5 (b) Generation of FM signal

d. Use the V_{in} found as the amplitude of the input signal and find Δf_{peak} on the oscilloscope. Compare Δf_{peak} calculated and measured. Display only the FM signal on the oscilloscope and trigger on the rising edge of the waveform such that you see the following Figure 5.6 (it will look like a ribbon):



$f_{max}=1/t_{min}$

$f_{min}=1/t_{max}$

$\Delta f_{peak}=(f_{max} - f_{min})/2$

Figure 5.6: Frequency components of the modulated signal

This "ribbon" displays all frequencies in the FM signal at once. The minimum and maximum frequencies can be easily detected and directly measured. Recall that Δf_{peak} is only half of the peek-to-peak frequency swing. What parameters determine the bandwidth of an FM signal?

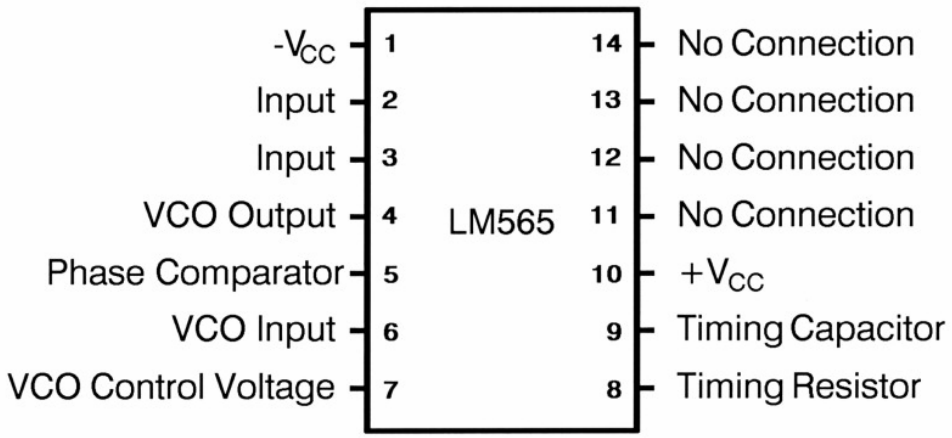
e. View the frequency domain waveform to obtain modulation indices of 2.4, 5.52, 8.65. These are zero carrier amplitude indices. Include calculations to verify your results in your lab report.

7.2 Demodulation

a. Build the FM demodulator utilizing the LM565N PLL shown in Figure 5.7. Connect the output of the FM modulator shown in Figure 5.5 (b) to the input of the FM demodulator.

b. Apply a sinusoidal message signal and observe the demodulated message signal. Sketch both waveforms. How do they compare?
Hint for the Tektronix Oscilloscope:
An FM demodulator is typically followed by a filter to block the carrier. Setting the oscilloscope Trigger Coupling to HF REJ will help sync on the modulation frequency.

c. Apply a square wave message signal and observe the demodulated message signal. Sketch both waveforms. How do they compare? Why is the demodulated sinusoidal message more faithfully reproduced than the demodulated square wave?



Phase Lock Loop Pin Assignment

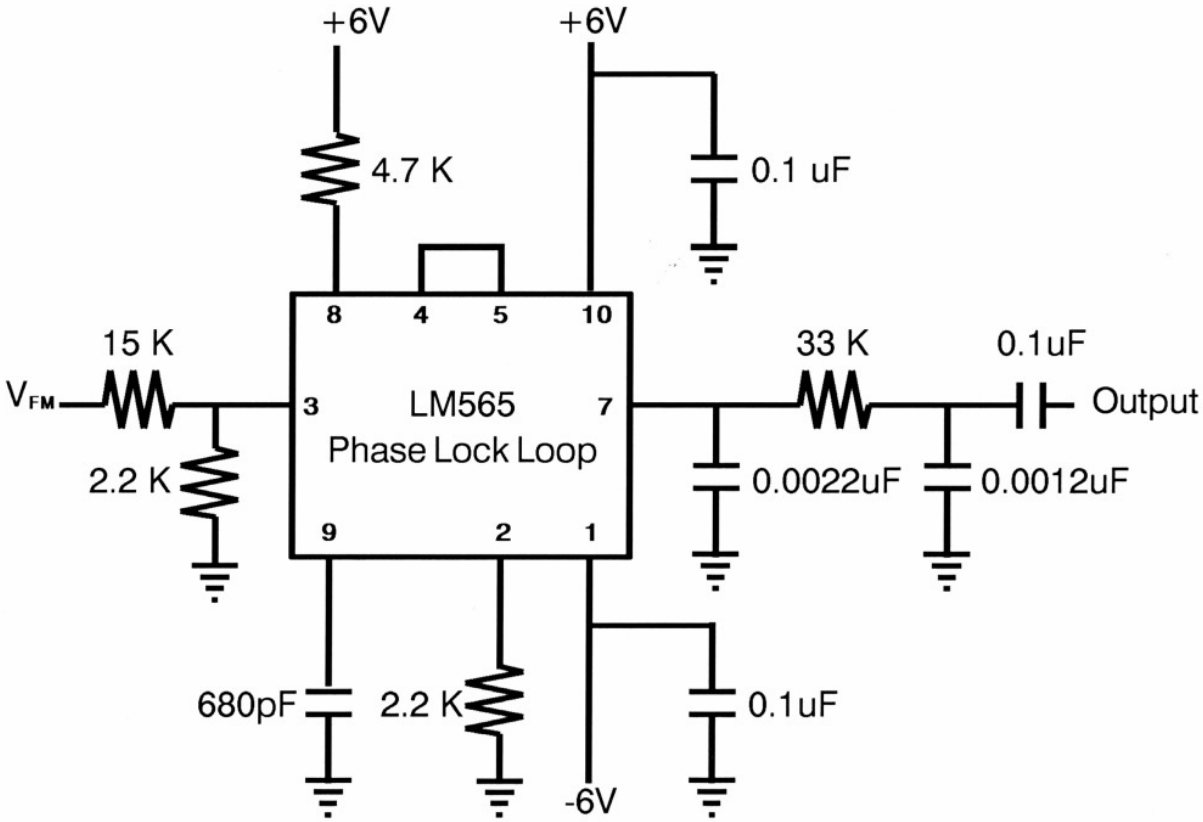


Figure 5.7. PLL as an FM Demodulator

Table 5.1

Bessel Functions of the first kind, $J_n(x)$

x	J_0	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9	J_{10}
0.0	1.00										
0.2	0.99	0.10									
0.4	0.96	0.20	0.02								
0.6	0.91	0.29	0.04								
0.8	0.85	0.37	0.08	0.01							
1.0	0.77	0.44	0.11	0.02							
1.2	0.67	0.50	0.16	0.03	0.01						
1.4	0.57	0.54	0.21	0.05	0.01						
1.6	0.46	0.57	0.26	0.07	0.01						
1.8	0.34	0.58	0.31	0.10	0.02						
2.0	0.22	0.58	0.35	0.13	0.03	0.01					
2.2	0.11	0.56	0.40	0.16	0.05	0.01					
2.4	0.00	0.52	0.43	0.20	0.06	0.02					
2.6	-0.10	0.47	0.46	0.24	0.08	0.02	0.01				
2.8	-0.19	0.41	0.48	0.27	0.11	0.03	0.01				
3.0	-0.26	0.34	0.49	0.31	0.13	0.04	0.01				
3.2	-0.32	0.26	0.48	0.34	0.16	0.06	0.02				
3.4	-0.36	0.18	0.47	0.37	0.19	0.07	0.02	0.01			
3.6	-0.39	0.10	0.44	0.40	0.22	0.09	0.03	0.01			
3.8	-0.40	0.01	0.41	0.42	0.25	0.11	0.04	0.01			
4.0	-0.40	-0.07	0.36	0.43	0.28	0.13	0.05	0.02			
4.2	-0.38	-0.14	0.31	0.43	0.31	0.16	0.06	0.02	0.01		
4.4	-0.34	-0.20	0.25	0.43	0.34	0.18	0.08	0.03	0.01		
4.6	-0.30	-0.26	0.18	0.42	0.36	0.21	0.09	0.03	0.01		
4.8	-0.24	-0.30	0.12	0.40	0.38	0.23	0.11	0.04	0.01		
5.0	-0.18	-0.33	0.05	0.36	0.39	0.26	0.13	0.05	0.02	0.01	
5.2	-0.11	-0.34	-0.02	0.33	0.40	0.29	0.15	0.07	0.02	0.01	
5.4	-0.04	-0.35	-0.09	0.28	0.40	0.31	0.18	0.08	0.03	0.01	
5.6	0.03	-0.33	-0.15	0.23	0.39	0.33	0.20	0.09	0.04	0.01	
5.8	0.09	-0.31	-0.20	0.17	0.38	0.35	0.22	0.11	0.05	0.02	0.01